Volterra Integro-Differential Equations: Models and Numerical Solution

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***Abstract*— Integro-differential equation governs several important real life and numerous scientific and engineering domains. A brief survey of the properties and different treatments of the one-dimensional integro-differentia problems is presented, and a numerical method is stated for solving some of this problem via many numerical methods. In this paper, the problem is considered as a linear Volterra integro-differential equation and solutions are obtained using the Nyström method decomposition with difference finites method. Once more, the integro-differential equation has been transformed into a numerically-solvable linear algebraic system and the Nyström numerical method simplifies the integral part in the equation. In order to demonstrate the method's accuracy and efficacy, certain numerical problems are resolved.**

***Keywords*— Integro-differential equation, Volterra models, Difference finites method, Quadrature method, Approximation solution.**

I. INTRODUCTION

Mathematical equations known as: Integro-Differential Equations (IDEs) combine integrals and derivatives. They appear in a variety of scientific and engineering domains, such as physics, biology, economics, and finance, where systems display memory- or history-dependent behavior. Integro-Deferential Equations (IDEs) incorporate the influence of previous values of the unknown function through the integration term, in contrast to Ordinary Differential Equations (ODEs), which solely involve derivatives. Problems involving diffusion, wave propagation, population dynamics, control theory, and other topics frequently include integral-differential equations. They offer a more accurate explanation of events that show memory effects or spatial interactions. Since the majority of IDEs cannot be resolved analytically, research has concentrated on creating numerical techniques to achieve approximations of solutions [1] .Many methods for determining the numerical solution of integro-differential equations, include the Adomain decompositions method by [2, 3], Collocation method by [4, 5, 6, 7], Hybrid linear multistep method [8, 9], [10] provided a computationally effective approach to solving Volterra integro-differential equations by using Legendre as a basis function for the solution of the integro-differential equations. In this paper, we research the computational solutions of a certain type of integro-differentials problems with continuous kernel.

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Volterra was the first person to describe the nonlinear Volterra integro-differential equation. It is used in a lot of different physical applications such as glass-forming process, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating. Literature in the fields of physics, biology, and engineering applications can provide additional information about the origins of these equations, for more details, see, [11]

This work considers linear and nonlinear Volterra integro-differential equations of the type

$$\varphi^{(\alpha)}(t) = f(t) + \int_0^t k(t,x)\big(\varphi(x)\big)^m dx, \quad 0 \le x,t \le 1, \qquad \alpha = 1, \qquad m \ge 1 \quad (1)$$

subject to initial condition

$$\varphi^{(i)}(0) = q_i, i = 0,1,\dots,N.$$

where $k(t,x)$ is the Fredholm integral kernel function, $f(t)$ is the known function, and $\varphi(t)$ is an unknown function to be determined.

II. SOME VOLTERRA INTEGRO-DIFFERENTIAL MODELS

I. Integro-differential model population growth:

In this section we will study the Volterra model for population growth of a species within a closed system. The nonlinear Volterra integro-differential equation can be used to describe the Volterra population model of the form

$$\dot{\varphi}(T) = a\varphi - b\varphi^2 - c\varphi \int_0^T (x)dx, \quad \varphi(0) = \varphi_0 \qquad (2)$$

Where $\varphi(T)$ denotes the population at time T , a, b, and c are constants and positive parameters, $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, $c > 0$ is the toxicity coefficient, and φ_0 is the initial population. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long run. When $b = 0$ and $c = 0$, Equation (2) becomes the Malthus differential equation

$$\dot{\varphi}(T) = a\varphi, \quad \varphi(0) = \varphi_0$$

That is assumed in the Malthus equation, the population growth is proportional to the current population. Several computational and analytical methods have been applied to find closed form solution and numerical approximations to the population model of the Volterra (2).

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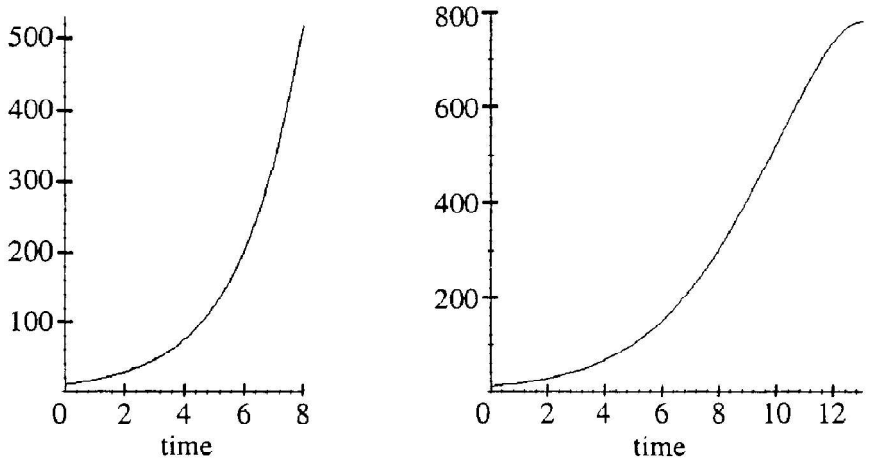

Fig. 1 [12] A: Exponential population evolution under constant values: **a=0.5, b=0, c=0, $\varphi_{0=10}$.**
B: Logistic population evolution under constant values: **a=0.5, b=0.001, c=0, $\varphi_{0=10}$.**

For further studies on the Volterra's population model, we refer to Wazwaz [11], Bordehor et al.[12], and see [12,14,15].

II. The Thomas-Fermi Equation

Thomas (1927) and Fermi (1928) independently developed the Thomas-Fermi equation to investigate the electron distribution and potentials of an atom. This equation has an essential importance in the field of mathematical physics. It was introduced first to study the multi-electron atoms. It was used in the description of the charge density in atoms of high atomic number. The Thomas-Fermi equation was also used to address the molecular theory, solid state theory, and hydrodynamic codes [11]. One example of a nonlinear Volterra integro-differential equation of the second class is the dimensionless Thomas-Fermi equation, which may be expressed as

$$\dot{\varphi}(T) = B + \int_0^T (x)^{\frac{-1}{2}} \varphi^{\frac{3}{2}}(x) dx,$$

For an isolated atom, the boundary conditions are given by

$$\varphi(0) = 1, \quad \lim_{T\to\infty} \varphi(T) = 0.$$

Notice that the potential $u\_(0) = B$ is important that will be the focus of this study.

For further studies on the Thomas-Fermi Equation, we refer to Wazwaz [11].

In the next, we will present and apply effective method to determine an approximation solution of the Volterra integro-differential equation. This method is the quadrature method decomposition with difference finites method.

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III. Basic definitions

The hereditary influences were examined by Volterra during the examination of a population growth model. The study led to a certain topic where differential and integral operators were used together in the same equation. This new type of equations was termed as Volterra integro-differential equations given in the form

$$\varphi^{(n)}(t) = f(t) + \int_0^t k\big(t, x, \varphi(x)\big)dx,$$

At least one of the derivatives $\varphi', \varphi'', \dots, \varphi^{(n)}$ must be present outside the integral sign in any Volterra integro-differential equation. The Volterra integro-differential equations may be observed when we convert an initial value problem to an integral equation by using Leibnitz rule.

The Volterra integro-differential equation appeared after its establishment by Volterra. It then appeared in many physical applications such as glass forming process, nanohydro-dynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert. More details about the sources where these equations arise can be found in physics, biology and engineering applications books, for example, see [11].

To determine a solution for the integro-differential equation, the initial conditions should be given, and this may be clearly seen as a result of involving $\varphi(t)$ and its derivatives. The initial conditions are needed to determine the exact solution.

IV. Finite difference approximations

A finite difference method proceeds by replacing the derivatives in the differential equations by finite difference approximations. This gives a large algebraic system of equations to be solved in place of the differential equation, something that is easily solved on a computer.

For the sake of simplicity, we shall consider the one-dimensional case only. The main concept behind any finite difference scheme is related to the definition of the derivative of a smooth function φ at a point $t \in \mathbb{R}$:

$$\varphi'(t) = \lim_{h \to 0} \frac{\ddot{o}(t+h) - \varphi(t)}{h},$$

and to the fact that when h tends to 0 (without vanishing), the quotient on the right-hand side provides a ”good” approximation of the derivative. This means that h should be small enough to get a good estimate. Actually, the approximation is good when the error commited in this approximation (*i.e.*

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when replacing the derivative by the differential quotient) tends towards zero when *h* tends to zero. If the function φ is sufficiently smooth in the neighborhood of *t*, it is possible to quantify this error using a Taylor expansion.

Definition (Taylor series):

Suppose the function φ is C^2continuous in the neighborhood of t. For any h > 0 we have:

$$\varphi(t+h) = \varphi(t) + h\,\ddot{o}'(t) + \frac{h^2}{2}\varphi''(t+\varepsilon),$$

where ε is a number between 0 and h (i.e. x+ε is point of]x, x+h[). For the treatment of problems, it is convenient to retain only the first two terms of the previous expression:

$$\varphi(t+h) = \varphi(t) + h\varphi'(t) + O(h^2),$$

where the term $O(h^2)$ indicates that the error of the approximation is proportional to h^2. This estimate is referred to as the forward difference approximant of φ'.

Likewise, we can define the first order backward difference approximation of φ'at point t as:

$$\varphi(t-h) = \varphi(t) - h\varphi'(t) + O(h^2).$$

Obviously, other approximations can be considered. In order to improve the accuracy of the approximation, we define a consistant approximation, called the central difference approximation, by taking the points t − h and t + h into account. Suppose that the function φ is three times differentiable in the vicinity of *t*:

$$\frac{\varphi(t-h) - \varphi(t-h)}{2h} = \varphi'(t) + O(h^2)$$

For more information on the finite difference method, see for example [16].

V. Numerical integration

The most popular approaches to numerical integration are detailed below. The first dimension of integrals over bounded intervals will be our main objective.

1) Quadrature Formula:

   Let f be a real function over the interval [a, b]. Computation directly the definite integral

$$I(f) = \int_a^b f(t)dt,$$

   might be complex and sometimes impossible. Any explicit formula that is suitable for providing an approximation of I(f) is said to be a quadrature formula or numerical integration formula.

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An example can be obtained by replacing f with an approximation f_n, depending on the integer $n \geq 0$, then computing $I(f_n)$instead of I(f). Letting

$$I_n(f) = I(f_n),$$

we have

$$I_n(f) = \int_a^b f_n(x)dx, \quad n \geq 0.$$

If $f \in C^0([a,b])$, the quadrature errors $E_n(f) = I(f) - I_n(f)$ satisfies

$$|E_n(f)| \leq \int_a^b |f(x) - f_n(x)|dx \leq (b-a)\|f - f_n\|_\infty.$$

We shall describe the quadrature or Nyström method for the approximate solutions of linear and nonlinear integro-differential equations of the second kind with continuous kernels. The goal of the quadrature methods is to approximate the definite integral of f(t) over the interval G=[a,b] by evaluating f(t) at a finite number of sample points. In this work, we will consider closed Newton-Cotes methods only (Simpson's rule).

Suppose that $a = t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)} = b$. A formula of the form

$$Q_n(f) = \sum_{i=1}^{n} w_i^{(n)} f\left(t_i^{(n)}\right),$$

with a property that

$$\int_a^b f(t)dt = Q_n(f) + E(f),$$

is called a numerical integration or quadrature formula. The term E(f) is called the truncation error for integration. The values $t_i^{(n)}$ are called the quadrature nodes and $w_i^{(n)}$are called the weights. A sequence $Q_n(f)$of quadrature formulas is called convergent if

$$Q_n(f) \to Q(f), \quad n \to \infty, \qquad \forall f \in C(G),$$

i.e., if the sequence of linear functionals (Q_n) converges pointwise to the integral Q.

The significance of the next two results is to understand that the error terms for the composite trapezoidal rule and composite Simpson rule are of the order $O(h^2)$ and $O(h^4)$, respectively. This shows that the error for Simpson's rule converges to zero faster than the error for the trapezoidal rule as the step size h decreases to zero.

Corollary 1. (Trapezoidal rule: Error analysis)**.** Suppose that G is subdivided into n subintervals $[x_i, x_{i+1}]$ of width $h = \frac{b-a}{n}$. The composite trapezoidal rule

$$T(f,h) = \frac{h}{2}\big(f(a) + f(b)\big) + h\sum_{i=1}^{n-1} f(x_i),$$

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is an approximation to the integral

$$\int_a^b f(t)dt = T(f,h) + E_T(f,h),$$

Furthermore, if $f \in C^2([a,b])$, there exists a value c with a<c<b so that the error term $E_T(f,h)$,has the form

$$E_T(f,h) = \frac{(a-b)f^{(2)}(c)}{12}h^2 = O(h^2).$$

Corollary 2. (Simpson's rule: Error analysis) [10]. Suppose that G is subdivided into 2n subintervals $[x_i, x_{i+1}]$ of width $h = \frac{b-a}{n}$. The composite Simpson rule

$$S(f,h) = \frac{h}{3}\big(f(a) + f(b)\big) + \frac{2h}{3}\sum_{i=1}^{n-1} f(x_{2i}) + \frac{4h}{3}\sum_{i=1}^{n} f(x_i),$$

is an approximation to the integral

$$\int_a^b f(t)dt = S(f,h) + E_s(f,h),$$

Furthermore, if $f \in C^4([a,b])$ there exists a value c with a<c<b so that the error term $E_s(f,h)$ has the form

$$E_S(f,h) = \frac{(a-b)f^{(4)}(c)}{180}h^4 = O(h^4).$$

VI. Computational Methods For Volterra integro-differential equation

For Urysohn integro-differential equations with continuous kernels, perhaps the most natural and well-known approximation technique is the Nyström method. That is, if $\varphi(t)$ satisfies

$$\varphi'(t) = K_1(t)\varphi(t) + F(t) + \int_0^t K\big(t,x,\varphi(x)\big)dx \tag{3}$$

then one can obtain an approximation to $\varphi(t)$ by replacing

$$\int_0^t K\big(t,x,\varphi(x)\big)dx = \sum_{j=1}^{n} K\left(t,x_j,\varphi_j(x)\right)w_i^{(n)}, \tag{4}$$

with quadrature points (nodes) $t_i^{(n)}$ contained in G and real quadrature weights $w_i^{(n)}$. Substituting the right-hand side of Eq(4) for the integral in Eq(4) generates the functional equation

$$\varphi'_n(t) + \sum_{j=1}^{n} K\left(t,x_j,\varphi_n(x_j)\right)w_i^{(n)} = F(t) \tag{5}$$

Where $\varphi_n(t)$ is an approximation to $\varphi(t)$. All the conventional quadrature rules can be written in the form of equation (3). The trapezoidal and Simpson rules are special cases of (4).

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A. For linear Volterra integro-differential equation:

Volterra integro-differential equation of the second kind of the following form

$$\varphi'(t) = K_1(t)\varphi(t) + F(t) + \int_0^t K(t,x)\varphi(x)dx \quad (6)$$

Now, we can apply Simpson method (see, for example [17]) for equation Eq. (6).

By the numerical integration formulas of modified Simpson method and finite difference formulation for the integro-differential equation Eq. (4), we obtain the following iteration formula

$$\varphi'(t_{2j}) = \frac{h}{3}\left(\sum_{i=0}^{j-1}\left(K(t_{2j},x_{2i})\varphi(t_{2i}) + K(t_{2j},x_{2i+1})\varphi(t_{2\_+1}) + K(\text{匆}_{2j},x_{2i+2})\varphi(t_{2i+2})\right)\right) + K_1(t_{2j})\varphi(t_{2j}) + F(t_{2j}),$$

we approximate $\varphi'(t_{2j})$ and $\varphi(t_{2i+1})$ by $\frac{\varphi_{2j+2}-\varphi_{2j}}{2h}$ and $\frac{\varphi_{2i+2}+\varphi_{2i}}{2}$ respectively. The last equation becomes

$$\begin{aligned}\varphi(t_{2j+2}) = {} & F(t_{2j}) \\ & + \frac{2h^2}{3}\left(\sum_{i=0}^{j-1}\left(K(t_{2j},x_{2i}) + 2K(t_{2j},x_{2i+1})\right)\varphi(t_{2i})\right. \\ & \left. + \sum_{i=0}^{j-1}\left(2K(t_{2j},x_{2i+1}) + K(t_{2j},x_{2i+2})\right)\ddot{o}(t_{2i+2})\right) \\ & + (2hK_1(t_{2j}) + 1)\varphi(t_{2j}) \end{aligned} \quad (7)$$

By recurrence, we can to calculate the approximation solutions φ of the equation Eq. (6) in all points t_{2j} for j=0,1,2,…,n.

B. For nonlinear Volterra integro-differential equation

Non linear Volterra integro-differential equation of the second kind of the following form

$$\varphi'(t) = K_1(t)\varphi(t) + F(t) + \int_0^t K(t,x,\varphi(x))dx \quad (8)$$

Now, we can apply Simpson method (see, for example [17]) for equation Eq. (8).

Now, by the numerical integration formulas of modified Simpson method and finite difference formulation for the integro-differential equation Eq. (8), we obtain the following iteration formula

$$\varphi'(t_{2j}) = \frac{h}{3}\left(\sum_{i=0}^{j-1}\left(K\left(t_{2j},x_{2i},\varphi(t_{2i})\right) + K\left(t_{2j},x_{2\,\text{嬰}+1},\varphi(t_{2i+1})\right) + K\left(t_{2j},x_{2i+2},\varphi(t_{2i+2})\right)\right)\right) + K_1(t_{2j})\varphi(t_{2j}) + F(t_{2j}),$$

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we approximate $\varphi'(t_{2j})$ and $\varphi(t_{2i+1})$ by $\frac{\varphi_{2j+2}-\varphi_{2j}}{2h}$ and $\frac{\varphi_{2i+2}+\varphi_{2i}}{2}$ respectively. The last equation becomes

$$\varphi(t_{2j+2}) = \frac{2h^2}{3}\left(\sum_{i=0}^{j-1}\left(K\left(t_{2j}, x_{2i}, \varphi(t_{2i})\right) + K\left(t_{2j}, x_{2i+1}, \varphi(t_{2i+1})\right) + K\left(t_{2j}, x_{2i+2}, \varphi(t_{2i+2})\right)\right)\right) + \left(2hK_1(t_{2j}) + 1\right)\varphi(t_{2j}) + 2hF(t_{2j}), \quad (9)$$

By recurrence, we can to calculate the approximation solutions φ of the equation Eq. (8) in all points t_{2j} for j=0,1,2,…,n.

For great detail and more information about the topic of nonlinear integro-differential equation and their numerical solution, see[1-9] and [17].

VII. Numerical examples

The quadrature method decomposition with difference finites method for solving the second kind Volterra integro-differential equations will be illustrated by studying the following four examples. The selected equations are linear and of order one.

Example 01 [11]: First we consider the linear Volterra integro-differential equations of the second kind of the form:

$$\varphi'(t) = 1 - 2t\sin t + \int_0^t \varphi(x)dx \quad (10)$$

with the initial condition given as $\varphi(0) = 0$, the exact solution is

$$\varphi(t) = t\cos(t).$$

We shall demonstrate the numerical resultants obtained by Nyström numerical methods to validate their application to integro-differential equation (10), and we take N=20,40, 60.

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N=20	Exact Solution	Approximation Solution	Absolute Error
0	0.0000e+00	0.0000e+00	0.0000e+00
0.1	9.9500e-02	1.0006e-01	5.6250e-04
0.2	1.9601e-01	1.9838e-01	2.3694e-03
0.3	2.8660e-01	2.9200e-01	5.4013e-03
0.4	3.6842e-01	3.7805e-01	9.6245e-03
0.5	4.3879e-01	4.5378e-01	1.4991e-02
0.6	4.9520e-01	5.1664e-01	2.1441e-02
0.7	5.3539e-01	5.6429e-01	2.8903e-02
0.8	5.5737e-01	5.9466e-01	3.7293e-02
0.9	5.5945e-01	6.0597e-01	4.6521e-02
1	5.4030e-01	5.9679e-01	5.6490e-02

TABLE I: Exact solution, Approximation solutions and Absolute errors to integro-differential equation (10), for **N=20**.

N=40	Exact Solution	Approximation Solution	Absolute Error
0	0.0000e+00	0.0000e+00	0.0000e+00
0.1	9.9500e-02	9.9797e-02	2.9670e-04
0.2	1.9601e-01	1.9723e-01	1.2148e-03
0.3	2.8660e-01	2.8934e-01	2.7439e-03
0.4	3.6842e-01	3.7329e-01	4.8663e-03
0.5	4.3879e-01	4.4635e-01	7.5575e-03
0.6	4.9520e-01	5.0599e-01	1.0787e-02
0.7	5.3539e-01	5.4991e-01	1.4517e-02
0.8	5.5737e-01	5.7607e-01	1.8707e-02
0.9	5.5945e-01	5.8276e-01	2.3310e-02
1	5.4030e-01	5.6858e-01	2.8277e-02

TABLE III: Exact solution, Approximation solutions and Absolute errors to integro-differential equation (10), for **N=40**.

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N=60	Exact Solution	Approximation Solution	Absolute Error
0	0.0000e+00	0.0000e+00	0.0000e+00
0.1	9.9500e-02	9.9702e-02	2.0124e-04
0.2	1.9601e-01	1.9683e-01	8.1657e-04
0.3	2.8660e-01	2.8844e-01	1.8389e-03
0.4	3.6842e-01	3.7168e-01	3.2563e-03
0.5	4.3879e-01	4.4384e-01	5.0522e-03
0.6	4.9520e-01	5.0241e-01	7.2060e-03
0.7	5.3539e-01	5.4508e-01	9.6929e-03
0.8	5.5737e-01	5.6985e-01	1.2485e-02
0.9	5.5945e-01	5.7500e-01	1.5551e-02
1	5.4030e-01	5.5916e-01	1.8859e-02

TABLE IIIII: Exact solution, Approximation solutions and Absolute errors to integro-differential equation (10), for **N=60**.

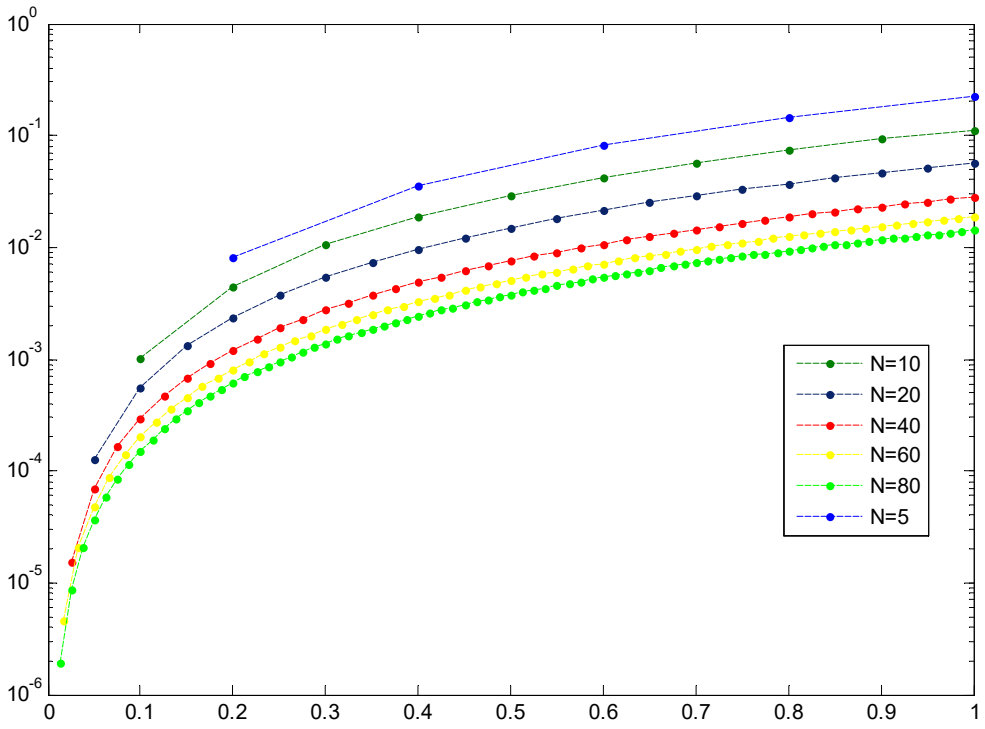

Fig. 2 Absolute errors to integro-differential equation (10); by using the quadrature method decomposition with difference finites method.

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Example 02 [11] : We consider the linear Volterra integro-differential equations of the second kind of the form:

$$\varphi'(t) = 1 + \int_0^t \varphi(x)dx \tag{11}$$

with the initial condition given as $\varphi(0) = 1$, the exact solution is

$$\varphi(t) = e^t.$$

We shall demonstrate the numerical resultants obtained by Nyström numerical methods to validate their application to integro-differential equation (11), and we take N=20, 40, 60.

N=20	Exact Solution	Approximation Solution	Absolute Error
0	1.0000	1.0000	0.0000e+00
0.1	1.1052	1.1078	2.6610e-03
0.2	1.2214	1.2270	5.6372e-03
0.3	1.3499	1.3588	8.9889e-03
0.4	1.4918	1.5046	1.2783e-02
0.5	1.6487	1.6658	1.7096e-02
0.6	1.8221	1.8441	2.2012e-02
0.7	2.0138	2.0414	2.7625e-02
0.8	2.2255	2.2596	3.4043e-02
0.9	2.4596	2.5010	4.1385e-02
1	2.7183	2.7681	4.9789e-02

TABLE IV: Exact solution, Approximation solutions and Absolute errors to integro-differential equation (11), for **N=20**.

N=40	Exact Solution	Approximation Solution	Absolute Error
0	1.0000	1.0000	0.0000e+00
0.1	1.1052	1.1065	1.3235e-03
0.2	1.2214	1.2242	2.8015e-03
0.3	1.3499	1.3543	4.9119e-03
0.4	1.4918	1.4982	6.8515e-03

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0.5	1.6487	1.6572	9.0554e-03
0.6	1.8221	1.8330	1.1566e-02
0.7	2.0138	2.0274	1.3680e-02
0.8	2.2255	2.2424	1.6847e-02
0.9	2.4596	2.4801	2.0468e-02
1	2.7183	2.7429	2.4610e-02

TABLE VI: Exact solutions, Approximation solutions and Absolute errors to integro-differential equation (11), for **N=40**.

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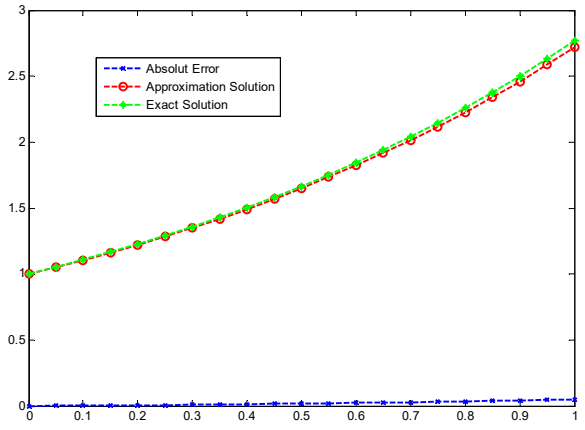
N=60	**Exact Solution**	**Approximation Solution**	**Absolute Error**
0	1.0000	1.0000	0.0000e+00
0.1	1.1052	1.1061	8.8081e-04
0.2	1.2214	1.2233	1.8640e-03
0.3	1.3499	1.3528	2.9692e-03
0.4	1.4918	1.4960	4.2184e-03
0.5	1.6487	1.6544	5.6362e-03
0.6	1.8221	1.8294	7.2501e-03
0.7	2.0138	2.0228	9.0909e-03
0.8	2.2255	2.2367	1.1193e-02
0.9	2.4596	2.4732	1.3597e-02
1	2.7183	2.7346	1.6345e-02

TABLE VIII: Exact solutions, Approximation solutions and Absolute errors to integro-differential equation (11),



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.Fig. II : Absolute errors to integro-differential equation (11); by using the quadrature method decomposition with difference finites method for **N=60**.

Example 03 [11]: We consider the linear Volterra integro-differential equations of the second kind of the form:

$$\varphi'(t) = 1 + t + \int_0^t (t - x)\varphi(x)dx \qquad (12)$$

with the initial condition given as $\varphi(0) = 1$, the exact solution is

$$\varphi(t) = e^t.$$

We shall demonstrate the numerical resultants obtained by Nyström numerical methods to validate their application to integro-differential equation (12), and we take N=10, 20, 60.

N=10	**Exact Solution**	**Approximation Solution**	**Absolute Error**
0	1.0000	1.0000	0.0000e+00
0.1	1.1052	1.0995	5.7041e-03
0.2	1.2214	1.2094	1.2006e-02
0.3	1.3499	1.3309	1.8973e-02
0.4	1.4918	1.4651	2.6685e-02
0.5	1.6487	1.6135	3.5238e-02
0.6	1.8221	1.7774	4.4742e-02
0.7	2.0138	1.9584	5.5324e-02
0.8	2.2255	2.1584	6.7131e-02
0.9	2.4596	2.3793	8.0332e-02
1	2.7183	2.6232	9.5118e-02

TABLE VII: Exacts solution, Approximation solutions and Absolute errors to integro-differential equation (12); for **N=10** by using the Nyström numerical method.

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N=20	Exact Solution	Approximation Solution	Absolute Error
0	1.0000	1.0000	0.0000e+00
0.1	1.1052	1.1024	2.7398e-03
0.2	1.2214	1.2156	5.7681e-03
0.3	1.3499	1.3407	9.1186e-03
0.4	1.4918	1.4790	1.2831e-02
0.5	1.6487	1.6318	1.6952e-02
0.6	1.8221	1.8006	2.1536e-02
0.7	2.0138	1.9871	2.6647e-02
0.8	2.2255	2.1932	3.2355e-02
0.9	2.4596	2.4209	3.8745e-02
1	2.7183	2.6724	4.5910e-02

TABLE VIIII: Exact solutions, Approximation solutions and Absolute errors to integro-differential equation (12); for **N=20** by using the Nyström numerical method.

N=60	Exact Solution	Approximation Solution	Absolute Error
0	1.0000	1.0000	0.0000e+00
0.1	1.1052	1.1043	8.8865e-04
0.2	1.2214	1.2195	1.8712e-03
0.3	1.3499	1.3469	2.9587e-03
0.4	1.4918	1.4877	4.1643e-03
0.5	1.6487	1.6432	5.5034e-03
0.6	1.8221	1.8151	6.9941e-03
0.7	2.0138	2.0051	8.6570e-03
0.8	2.2255	2.2150	1.0516e-02
0.9	2.4596	2.4470	1.2598e-02
1	2.7183	2.7033	1.4935e-02

TABLE IXII: Exact solutions, Approximation solutions and Absolute errors to integro-differential equation (12); for **N=60** by using the Nyström numerical method

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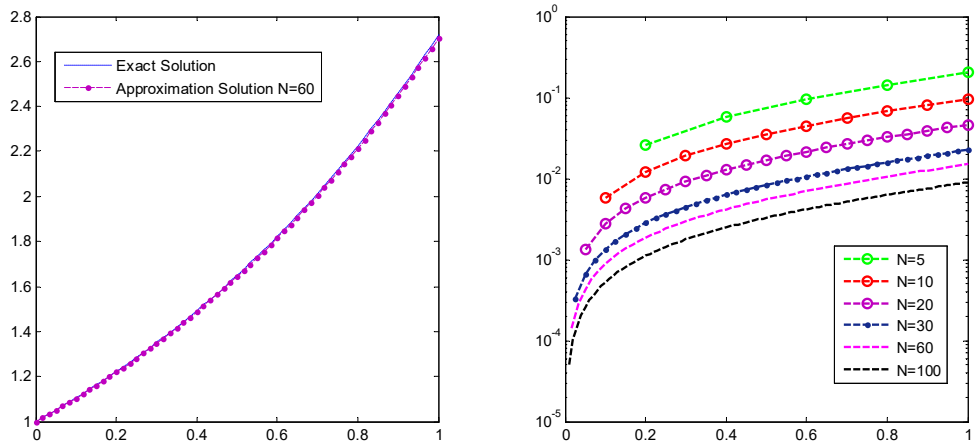

Fig. III: Absolute errors to integro-differential equation (12); by using the quadrature method decomposition with difference finites method.

Example 04: We consider the linear Volterra integro-differential equations of the second kind of the form:

$$\varphi'(t) = \frac{-t^2}{2}\varphi(t) + \int_0^t (t-x)\varphi(x)dx \qquad (13)$$

with the initial condition given as $\varphi(0) = 1$, the exact solution is

$$\varphi(t) = 1.$$

We shall demonstrate the numerical resultants obtained by Nyström numerical methods to validate their application to integro-differential equation (13), and we take N=10, 20, 40.

N=10	Exact Solution	Approximation Solution	Absolute Error
0	1.0000	1.0000e+00	0.0000e+00
0.1	1.0000	9.9950e-01	4.9983e-04
0.2	1.0000	9.9900e-01	9.9925e-04
0.3	1.0000	9.9850e-01	1.4973e-03
0.4	1.0000	9.9801e-01	1.9918e-03
0.5	1.0000	9.9752e-01	2.4801e-03
0.6	1.0000	9.9704e-01	2.9581e-03
0.7	1.0000	9.9658e-01	3.4211e-03
0.8	1.0000	9.9614e-01	3.8639e-03
0.9	1.0000	9.9572e-01	4.2806e-03

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1	1.0000	9.9533e-01	4.6651e-03

TABLE X: Exact solutions, Approximation solutions and Absolute errors to integro-differential equation (13); for **N=10** by using the Nyström numerical methods.

N=20	Exact Solution	Approximation Solution	Absolute Error
0	1.0000	1.0000e+00	0.0000e+00
0.1	1.0000	9.9997e-01	3.1248e-05
0.2	1.0000	9.9994e-01	6.2466e-05
0.3	1.0000	9.9991e-01	9.3568e-05
0.4	1.0000	9.9988e-01	1.2441e-04
0.5	1.0000	9.9985e-01	1.5478e-04
0.6	1.0000	9.9982e-01	1.8442e-04
0.7	1.0000	9.9979e-01	2.1303e-04
0.8	1.0000	9.9976e-01	2.4026e-04
0.9	1.0000	9.9973e-01	2.6575e-04
1	1.0000	9.9971e-01	2.8914e-04

TABLE XII: Exact solutions, Approximation solutions and Absolute errors to integro-differential equation (13); for **N=20** by using the Nyström numerical methods.

N=60	Exact Solution	Approximation Solution	Absolute Error
0	1.0000	1.0000e+00	0.0000e+00
0.1	1.0000	1.0000e+00	4.9996e-06
0.2	1.0000	9.9999e-01	9.4951e-06
0.3	1.0000	9.9999e-01	1.3976e-05
0.4	1.0000	9.9998e-01	1.8918e-05
0.5	1.0000	9.9998e-01	2.3789e-05
0.6	1.0000	9.9997e-01	2.8550e-05
0.7	1.0000	9.9997e-01	3.3151e-05
0.8	1.0000	9.9996e-01	3.7541e-05
0.9	1.0000	9.9996e-01	4.1660e-05
1	1.0000	9.9995e-01	4.6166e-05

TABLE XIIII: Exact solutions, Approximation solutions and Absolute errors to integro-differential equation (13);

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for **N=60** by using the Nyström numerical methods.

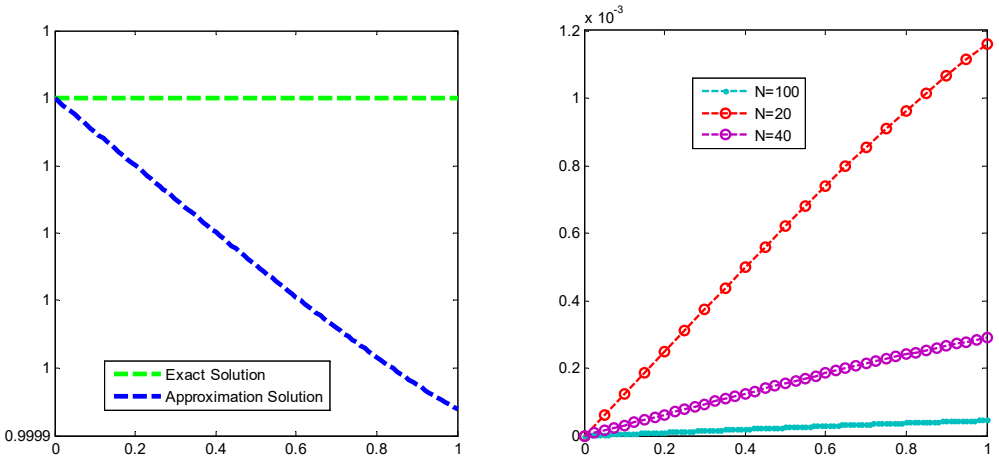

Fig. IV: Absolute errors to integro-differential equation (13); by using the quadrature method decomposition with difference finites method.

VIII. Conclusion

The present study involves the examination the quadrature method decomposition with difference finites method to solve the initial value problems for a kind of one-order linear Volterra integro-differential equation. Illustrative examples are included to demonstrate the validity and applicability of the method and the tables of results presented reveal that the absolute error decreases when the degree of approximation increases.

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