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ACO-based Constrained Optimal Control of Switched Linear Systems with Finite Time Horizon

Ahmed Anis Kahloul[1], Anis Sakly[2]

Electrical Department, National Engineering School of Monastir
University of Monastir
Skanes, Monastir, Tunisia
[1]kahloul_ahmedanis@yahoo.fr
[2]sakly_anis@yahoo.fr

Abstract— **Solving constrained optimal control problems of linear switched systems requires two main steps: determining the optimal control and finding optimal switching instants that minimize a well-defined functional cost. In this contest, this paper suggests a hybrid approach based on two main methods: Ant Colony Optimization (ACO) and Pontryagin Maximum Principle (PMP). This latter is associated to the Lagrange method in order to guarantee that the system respects the constraints imposed on it. To illustrate the proposed approach, a numerical example is provided by the end of this paper.**

Keywords— **Constrained Optimal Control, Linear Switched Systems, ACO, PMP, Lagrange multipliers.**

I. Introduction

Optimal control is one of the most important areas of extreme value theory. It has great practical applications in various fields of human activity [1] and in particular for switched systems. These latter, representing a particular class of hybrid systems, consist of a finite set of continuous subsystems with a switching law defining the active one at each time instant [2]–[5]. They make part of different real processes like chemical processes, flexible manufacturing systems, automotive industry, aircraft and air traffic control, large-scale power systems, computer controlled systems and communication networks [6]–[11].

The optimal control of switched systems has been an intriguing subject of study for researchers [7]–[13]. Its complexity consists in the necessity of determining not only the optimal input but also the optimal switching instants from a subsystem to another. However, the real aspect of the switched systems often requires to bear in mind the existence of constraints on the system, and one needs to strictly respect them while solving this kind of problems [14]–[18].

This paper suggests a hybrid approach to solve the optimal control problem of switched linear systems with pure state constraints in finite time horizon. In this approach, the problem is divided into two parts. In the first one, a conventional method based on PMP associated with the Lagrange method is used to determine an optimal control law. Satisfying the Karush-Kuhn-Tucker conditions (KKT) in this method insures the feasibility and the respect of the constraints put on the system. On the other hand, a metaheuristic is used to find the optimal switching instants that guarantee the minimization of a well-defined performance criterion. The advantage of this hybrid approach is boosting the convergence of the functional cost to a global optimum.

By the end of the article, a numerical example is given to illustrate the effectiveness of the suggested approach.

II. Problem Formulation

A. Optimal control

Switched linear systems are described by a finite number M of subsystems as follow:

$$\dot{x} = A_i x + B_i u \qquad (1)$$

where:

$$i \in I = \{1,2,\dots,M\} \qquad (2)$$

The switch from a subsystem to another is orchestrated by a switching sequence in $t \in [t_0, t_f]$ defined by:

$$\sigma = \big((t_0, i_0), (t_1, i_1), \dots, (t_k, i_k), \dots, (t_K, i_K)\big) \qquad (3)$$

At instant t_k, the system switches from subsystem i_{k-1} to i_k which is active during the time interval $[t_k, t_{k+1}]$. Note that $K \geq 0$; $t_0 \leq t_1 \leq \dots \leq t_K$ and $i_k \in I = \{1,2,\dots,K+1\}$ for $k = 1,2,\dots,K$.

The resolution of a state constrained optimal control problem of such systems consists in minimizing a defined performance criterion J:

$$J = \frac{1}{2} x(t_f)^t Q_f x(t_f) + M_f x(t_f) + W_f + \int_{t_0}^{t_f} \left(\frac{1}{2} x^t Q x + x^t V u + \frac{1}{2} u^t R u + M x + N u + W \right) \qquad (4)$$

while respecting the state constraints:

$$g(x) \leq 0 \qquad (5)$$

$$h(x) = 0 \qquad (6)$$

Vectors $g(x) = [g_1(x), \dots, g_q(x)]^t$ and $h(x) = [h_1(x), \dots, h_p(x)]^t$ represent respectively the q inequalities and the p equalities linear state constraints in the finite time interval $[t_0, t_f]$.

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The initial state is $x(t_0) = x_0$ and Q_f, M_f, M_f, Q, V, R, M, N and W are matrixes with appropriate dimensions. Q_f, $Q \geq 0$ and $R > 0$.

To minimize the functional cost given by (4), two steps are needed: to determine the optimal control law assuring the respect of constraints mentioned in (5) and (6), then to find the optimal switching instants from a subsystem to another.

For the first part of the problem, we start by constructing the Lagrangian:

$$\mathcal{L}(x,u) = \frac{1}{2}x^tQx + \frac{1}{2}x^tVu + \frac{1}{2}u^tRu + Mx + Nu + W + \lambda^t g + \mu^t h \quad (7)$$

where $\lambda = [\lambda_1, \dots, \lambda_q]$ and $\mu = [\mu_1, \dots, \mu_p]$ are the Lagrange multipliers satisfying the Karush-Kuhn-Tucker conditions.

KKT conditions:

Suppose that functions g and h are continuous on $\mathbb{R}$ and differentiable at x^*. If x^* is a local extreme, then there exist nonzero vectors of Langrange multipliers $\lambda = [\lambda_1, \dots, \lambda_q]$ and $\mu = [\mu_1, \dots, \mu_p]$ such that [1],[17]:

$$\nabla_x \mathcal{L}\,(x^*, u, \lambda, \mu, t) = 0 \quad (8)$$
$$\nabla_\lambda \mathcal{L}\,(x^*, u, \lambda, \mu, t) = g_j(x^*) \leq 0 \quad (9)$$
$$\nabla_\mu \mathcal{L}\,(x^*, u, \lambda, \mu, t) = h_l(x^*) = 0 \quad (10)$$
$$\lambda_j{}^* g_j(x^*) = 0 \quad (11)$$
$$\lambda^* \geq 0 \quad (12)$$
$$j = 1, \dots, q \text{ and } l = 1, \dots, p$$

Next, we construct the augmented Hamiltonian [16]

$$H = \frac{1}{2}x^tQx + x^tVu + \frac{1}{2}u^tRu + Mx + Nu + W + p^t(A_ix + B_iu) + \lambda^t g + \mu^t h \quad (13)$$

From the last equation, we write the state, the costate equations and the stationary condition for each active subsystem:

$$\frac{\partial H_i}{\partial p} = \dot{x} \quad (14)$$
$$-\frac{\partial H_i}{\partial x} = \dot{p} \quad (15)$$
$$\frac{\partial H_i}{\partial u} = 0 \quad (16)$$

B. Ant Colony switching instants Optimization

Ant Colony Optimization (ACO) is one of the most famous methods through biologically inspired heuristics. The first ACO algorithms were introduced in the early 1990's by Marco Dorigo and colleagues [19]-[21]. The fundamental thought of ACO is to imitate the cooperative behaviour of ant colonies [22].

Ants, while collecting food, always start by searching around their nest randomly. When an ant finds food, it takes it back to the nest and leaves, along its path, pheromone. This latter guides other ants to the source of the food, and its quantity depends on the quantity and quality of the food. Accordingly, pheromone represents an indirect way of communication between ants and it helps them find shortcuts between their nest and food sources. The artificial ant colonies are, thus, created after the real ones [23].

An ACO algorithm is based on the following equations:

- Pheromone update's equation

$$\tau_{ij} \leftarrow (1-\rho)\tau_{ij} + \textstyle\sum_{k=1}^{n} \Delta\tau_{ij}^{F} \quad (17)$$

- Probability of going to node j

$$p_{ij}^F = \frac{\tau_{ij}^\alpha \eta_{ij}^\beta}{\sum_{l\in j} \tau_{il}^\alpha \eta_{il}^\beta} \quad (18)$$

- Heuristic information

$$\eta_{ij} = \frac{1}{D_{ij}} \quad (19)$$

where:

$\boldsymbol{\rho}$: evaporation rate	$\boldsymbol{l}$: node not yet visited by any ant
$\boldsymbol{\Delta\tau_{ij}^F}$: quantity of pheromone laid on edge joining nodes i and j by ant F	$\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$: parameters controlling the relative importance of the pheromone
$\boldsymbol{n}$: number of ants	

To optimize switching instants for a constrained linear switched system optimal control, some conditions should be considered [24]:

- each path is composed by n_c nodes
- each node represents a switching instant
- the length of the path made an ant F, represents the cost of the distance between nodes i *and* j
- probability p_{ij}^k depends only on the pheromone quantity deposited between nodes i and j.

When the optimal control problem is solved (part (A)), the algorithm of hybrid approach evaluates within the search space $[t_0, t_f]$ each tour of ant formed by n_c nodes. It then finds the best tour corresponding to minimal cost for the edge nodes with high probability. Then, it updates the search space limited by the value of nodes found, and repeats again the assessment of the tours. Finally, the best sequence corresponding to minimal cost is returned by the algorithm.

III. NUMERICAL EXAMPLE

Consider a linear switched system consisting of two subsystems [14],[16],[25] – [27]:

- Subsystem 1:

$$\dot{x} = A_1x + B_1u = \begin{bmatrix} 0.6 & 1.2 \\ 0.8 & 3.4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \quad (20)$$

- Subsystem 2:

$$\dot{x} = A_2x + B_2u = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u \quad (21)$$

Subject to the state constraint:

$$x_2 \leq 1.5 \quad (22)$$

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Fixing the initial time $t_0 = 0s$ and the final time $t_f = 2s$, the system switches from subsystem 1 to subsystem 2 at $t = t_1$ $(t_0 \le t_1 \le t_f)$. Solving the constrained optimal control problem consists in finding the optimal switching instant t_1 that assures the minimization of the criterion J.

$$J = \frac{1}{2}[(x_1(t_f) - 4)^2 + (x_2(t_f) - 2)^2] + \frac{1}{2}\int_{t_0}^{t_f}((x_2(t) - 2)^2 + u^2(t))dt \quad (23)$$

The initial conditions are $x_1(0) = 0$ and $x_2(0) = 2$.

Using equation (7), we construct the Lagrangian:

$$\frac{1}{2}(x_2(t) - 2)^2 + \lambda(x_2(t) - 1.5) \quad (24)$$

The Lagrange multiplier $\lambda = 0.5$ is obtained by applying the KKT conditions in (8–12). So, the augmented Hamiltonian is:

- for $t \in [t_0, t_1)$

$$H_1 = \frac{1}{2}((x_2(t) - 2)^2 + u^2) + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}(A_1x + B_1u) + 0.5(x_2(t) - 1.5) \quad (25)$$

- for $t \in [t_1, t_f]$

$$H_2 = \frac{1}{2}((x_2(t) - 2)^2 + u^2) + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}(A_2x + B_2u) + 0.5(x_2(t) - 1.5) \quad (26)$$

After resolving the first stage of the problem, we search the optimal switching system using the ACO algorithm and numerical parameters of this latter are initialized:

- Number of ants $n = 40$;
- Proposed number of nodes for each switching sequences $n_p = 100$;
- $\alpha = 1$;
- $\rho = 0.99$.

The optimal switching instant obtained is $t_{1opt} = 1.56s$ and the corresponding optimal cost is $J_{opt} = 2.5439$. Figures 1, 2 and 3 show the evolution of the state and control with and without constraint.

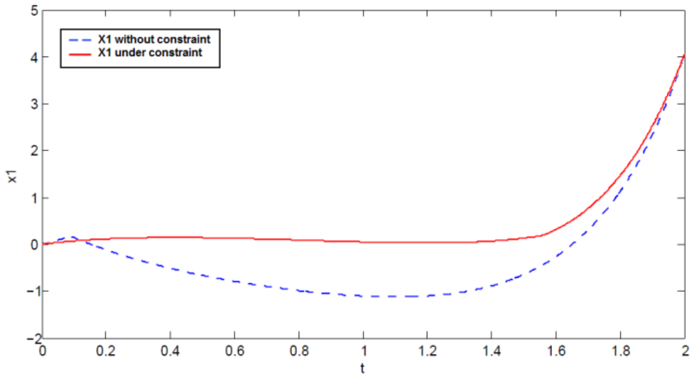

Fig. 1 x_1 evolution under and without constraint

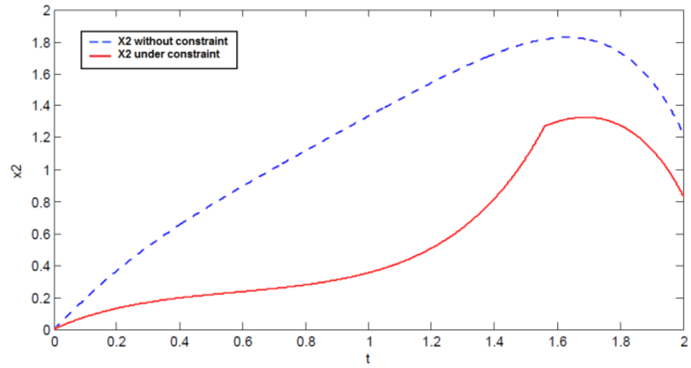

Fig. 2 x_2 evolution under and without constraint

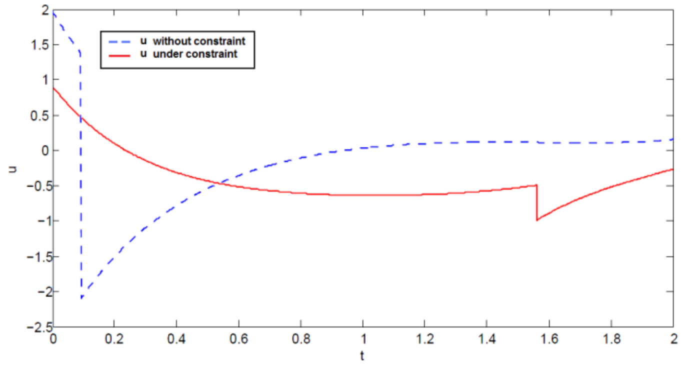

Fig. 2 u evolution under and without constraint

By applying ACO based hybrid approach to solve the constrained optimal control of this linear switched system, the state x_2 respects the (22) by remaining under the value 1.5. On the other hand, this constraint is never without an influence on the state x_1 and the signal input u as shown in the figures Fig. 2 and Fig. 3. However, obeying the constraint implies an elevation in the functional cost. In fact, the system cost rises from $J = 1.4434$ without constraints to reach the value $J_{opt} = 2.5439$ under constraints.

IV. CONCLUSIONS

The use of hybrid approaches in the solving the problems of constrained optimal control of switched systems has been rising during the last years. In fact, this article proposes an

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approach composed of a conventional method (PMP) and a metaheuristic one (ACO) to determine the optimal switching instants while respecting the constraints imposed on the system. The main advantage of this approach is to never stop at a local optimum in order to converge to a global optimum. However, it would be more interesting to use this approach on a more general scale by applying it on nonlinear switched system while respecting mixed constraints on the state and the control.

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