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Robust Stabilization of Interval Systems: A Compound Matrix Approach

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Abstract— **This paper is concerned with the robust stability and stabilization of systems with bounded parametric uncertainties. Robust stability conditions for such systems, referred to as interval systems, are derived based on compound matrices. Obtained results are then applied for robust stabilization. Numerical simulations are performed on an illustrative example to show the effectiveness of the proposed approach.**

***Index Terms*—Uncertain systems, interval uncertainty, robust stabilization, additive compound matrix, Lozinskiĭ measure.**

I. Introduction

The problems of stability and stabilization of systems with bounded parametric uncertainties, often called interval systems, are very common in robust control. Since the seminal work of Kharitonov [1] about interval polynomials, many results have been reported in literature. Several ones are extreme point results i.e. the stability of a family of linear interval plants is judged depending on a finite number of extreme plants [2, 3, 4, 5]. Principal adopted approaches are based on Kharitonov-like theory [6, 7, 8], interval arithmetic [9, 10,11, 12, 13], structured singular value analysis [14] and evolutionary methods [15, 16, 17, 18]. Most of the reported results are limited to stability analysis [19]. Many others, especially those using the Kharitonov theorem, are dealing only with systems described by transfer functions. In this paper, the robust stability study and the stabilization of linear time invariant systems is performed based on a compound matrix approach [20, 21, 22]. The paper is organized as follows. In section 2, the theoretical preliminaries about compound matrices are introduced. The proposed method for robust stability study and stabilization is exposed in section 3 and illustrated by a simulation example in section 4.

II. Preliminaries And Problem Formulation

A. Preliminaries and Notation

Let $M_n(\mathbb{R})$ be the space of *n*-square matrices with entries in $\mathbb{R}$ and let *A* be a matrix in $M_n(\mathbb{R})$ and *k* an integer in [1, *n*]. We note by $\wedge$ the exterior product in $\mathbb{R}^n$.

Definition 1 [20,21] The additive compound matrix $A^{[k]}$ of A, with respect to the canonical basis in the k^{th} exterior product space $\Lambda^k\mathbb{R}^n$ is a linear operator on $\Lambda^k\mathbb{R}^n$ and can be defined on a decomposable element $v_1 \wedge v_2 \wedge ... \wedge v_k$ by:

$$A^{[k]}(v_1 \wedge ... \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge ... \wedge Av_i \wedge ... \wedge v_k, \quad \forall v_1,...,v_k \in \mathbb{R}^n \tag{1}$$

Relations between entries (a_{ij}) of *A* and those of $A^{[k]}$ ($\tilde{a}_{ij}$) are linear.

Let *i* be an integer in $[1, C_n^k]$. If we note by (*i*) = ($i_1,...,i_k$) the i^{th} member in the lexicographic ordering of integer k-tuples such that $1 \le i_1 < ... < i_k \le n$, we can obtain the additive compound matrix entries from the following result.

Proposition 1 [20, 21]

$$\tilde{a}_{ij} = \begin{cases} a_{i_1 i_1} + ... + a_{i_k i_k}, & if\ (i) = (j), \\ (-1)^{r+s} a_{j_r i_s}, & if\ exactly\ one\ entry\ i_s\ of\ (i) \\ & does\ not\ occur\ in\ (j)\ and\ j_r \\ & does\ not\ occur\ in\ (i), \\ 0 & if\ (i)\ differs\ from\ (j)\ in\ two \\ & or\ more\ entries. \end{cases} \tag{2}$$

In particular, we have $A^{[1]} = A$, $A^{[n]} = trace(A)$ and for $A \in M_3(\mathbb{R})$, the second additive compound matrix is:

$$A^{[2]} = \begin{pmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22}+a_{33} \end{pmatrix} \tag{3}$$

Definition 2 [21] Let $|.|$ a vector norm on $M_n(\mathbb{R})$ and A a matrix in $M_n(\mathbb{R})$. The Lozinskiĭ measure (logarithmic measure) μ of A with respect to $|.|$ is defined by

$$\mu(A) = \lim_{h \to 0^+} \frac{|I + hA| - 1}{h} \tag{4}$$

As examples, Lozinskiĭ measure of a matrix A with respect to the three common vector norms (L_1, L_1 and L_1):

$|x|_1 = \sum_i |x_i|$, $|x|_2 = \sqrt{\sum_i |x_i|^2}$ and $|x|_\infty = \sup_i |x_i|$ are:

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$$\mu_1(A) = \sup_j (a_{jj} + \sum_{i, i\neq j} |a_{ij}|),$$
$$\mu_2(A) = s(\frac{A + A^T}{2}) \text{ and} \quad (5)$$
$$\mu_\infty(A) = \sup_i (a_{ii} + \sum_{j, j\neq i} |a_{ij}|).$$

Where $s(A)$ denotes the maximum real part of the eigenvalues of A.

Compound matrices represent, as illustrates the following result, an interesting formalism for the stability study of matrices.

A matrix $A \in M_n(\mathbb{R})$ is said to be stable (Hurwitz stable) if all its eigenvalues have strictly negative real parts.

Theorem 1 [21] if $(-1)^n \det(A) > 0$ then A is stable if and only if there exists a Lozinskii measure μ on $M_m(\mathbb{R})$ such that

$$\mu(A^{[2]}) < 0 \,,\; m = C_n^2 \,. \quad (6)$$

In this work, the same formalism is adopted in the case of interval systems. The following notations and basic interval arithmetic operations on $\mathbb{R}$ will be used in the sequel.

- The set of uncertain parameters $a \in [\underline{a}, \overline{a}]$ will be denoted by $[a]$.
- An interval matrix A with uncertain entries $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$ will be symbolized as $[A] = ([a_{ij}])_{1\le i, j\le n} = [\underline{A}, \overline{A}]$.
- The notation is the same for interval polynomials: $[P(x)] = \sum_{i=0}^{n} [a_i] x^i$
- For $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$ we have

$$\begin{aligned}
&[a]+[b] = [\underline{a}+\underline{b}, \overline{a}+\overline{b}]\\
&[a]-[b] = [\underline{a}-\overline{b}, \overline{a}-\underline{b}]\\
&[a].[b] = [\min(\underline{a}\underline{b}, \overline{a}\overline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}), \max(\underline{a}\underline{b}, \overline{a}\overline{b}, \underline{a}\overline{b}, \overline{a}\underline{b})]\\
&\|[a]\| = [\min(|\underline{a}|, |\overline{a}|), \max(|\underline{a}|, |\overline{a}|)]
\end{aligned} \quad (7)$$

Using the relations below, we define the determinant and the Lozinskiĭ measure of an interval matrix $[A] = ([a_{ij}])_{1\le i, j\le n}$ as follows:

$$\begin{aligned}
&\det([A]) = [\min(\det(A), A \in [\underline{A}, \overline{A}]), \max(\det(A), A \in [\underline{A}, \overline{A}])]\\
&\mu([A]) = [\min(\mu(A), A \in [\underline{A}, \overline{A}]), \max(\mu(A), A \in [\underline{A}, \overline{A}])]
\end{aligned} \quad (8)$$

We recall that an interval matrix $[A] = [\underline{A}, \overline{A}] \subset M_n(\mathbb{R})$ is said to be stable or "robustly stable" if all the matrices in $[\underline{A}, \overline{A}]$ are stable.

B. Problem Formulation

Consider a linear time-invariant (LTI) multivariable dynamic system described in the state space by

$$\dot{x} = [A]x + [B]u \quad (9)$$

where $x = x(t) \in \mathbb{R}^n$ and $u = u(t) \in \mathbb{R}^r$ are, respectively, the state and input vectors. $[A] \in \mathbb{R}^{n\times n}$ and $[B] \in \mathbb{R}^{n\times r}$ are interval state matrices. The pair $([A],[B])$ is assumed to be controllable for any fixed value of the uncertain parameters.

Define a linear state feedback control law as

$$u = -Kx, \quad K = (k_{ij})_{i=1..r,\, j=1..n} \quad (10)$$

The state space model of the closed loop system becomes

$$\dot{x} = [A_c]x \triangleq ([A] - [B]K)x \quad (11)$$

The robust stabilization problem consists on finding a real gain matrix that stabilizes the system (11) for all the possible values of its uncertain parameters.

III. Proposed Stabilization Method

The stability of system (9) is equivalent to the stability of the interval matrix $[A_c]$. So we focus our study on the stability of interval matrices using the compound matrix method. The following result is an immediate generalization of theorem 1.

Theorem 2 Let $[A]$ be an interval matrix on $M_n(\mathbb{R})$. If $(-1)^n \det([A]) > 0$ then $[A]$ is stable if and only if there exists a Lozinskiĭ measure μ on $M_m(\mathbb{R})$ such that

$$\mu([A]^{[2]}) < 0 \,,\; m = C_n^2 \,. \quad (12)$$

Proof Define $[A] = [\underline{A}, \overline{A}]$ and $A_1 \in [\underline{A}, \overline{A}]$ such that:

- $(-1)^n \det(A) > 0$ for every $A \in [\underline{A}, \overline{A}]$
- $\mu([A_1^{[2]}]) = \max(\mu(A),\, A \in [\underline{A}, \overline{A}])$

if $[A]$ is stable then so is A_1 and according to theorem 1, we have $\mu([A]^{[2]}) = \mu(A_1^{[2]}) < 0$. Reciprocally, if $\mu([A]^{[2]}) < 0$ then, for every $A \in [\underline{A}, \overline{A}]$, we have $\mu(A^{[2]}) \le \mu([A]^{[2]}) < 0$. This implies, by applying theorem 1, the stability of A and consequently the stability of $[A]$.

A more practical result is proposed in the next corollary using the Lozinskiĭ measure associated with the L_1 vector norm. Other sufficient stability conditions can be derived using different Lozinskiĭ measures.

Corollary 1 Let $[\tilde{A}] = (\tilde{a}_{ij})_{1\le i, j\le m}$ be the second additive compound matrix of $[A]$. If $(-1)^n \det([A]) > 0$ then

$$[A] \text{ is stable if } \;\overline{\tilde{a}}_{jj} + \sum_{i=1,\, i\neq j}^{m} \|[\tilde{a}_{ij}]\| < 0 \,,\; j = 1..m \,,\; m = C_n^2 \quad (13)$$

Proof Consider the Lozinskiĭ measure associated with $|\,.\,|_1$.

$\overline{\tilde{a}}_{jj} + \sum_{i=1,\, i\neq j}^{m} \|[\tilde{a}_{ij}]\| < 0$, $j = 1..m$ implies

$[\tilde{a}_{jj}] + \sum_{i=1,\, i\neq j}^{m} \|[\tilde{a}_{ij}]\| < 0$, $j = 1..m$ then

$$\mu_1([A]) = \sup_j ([\tilde{a}_{jj}] + \sum_{i, i\neq j}^{m} \|[\tilde{a}_{ij}]\|) < 0$$

and $[A]$ is consequently stable according to theorem 2.

The stabilization of system (11) can be achieved by applying corollary 1 to the matrix $[A_c]$.

A more explicit formulation is presented in the next result where the choice of the gain matrix K is conditioned by linear interval inequalities. Such inequalities can be transformed by minorization and majorization into simple linear inequalities.

Proposition 2 The pair $([A],[B])$ can be stabilized by the linear state feedback $u=-Kx,\ K=(k_{ij})_{i=1..r,\,j=1..n}$, if the following $m+1$ interval inequalities are satisfied:

$$(-1)^n \det([A]-[B]K) > 0 \tag{14}$$

$$\overline{\tilde{a}}_{jj}(K) + \sum_{i=1,\, i\neq j}^{m} \left|[\tilde{a}_{ij}(K)]\right| < 0,\ j=1..m,\ m=C_n^2 \tag{15}$$

where $[\tilde{a}_{ij}(K)] = [\underline{\tilde{a}}_{ij}(K), \overline{\tilde{a}}_{ij}(K)]$ are the entries of the second additive compound matrix of $([A]-[B]K)$.

IV. NUMERICAL EXAMPLE

Consider the problem of stabilizing the longitudinal motion speed of a helicopter modeled by [23]:

$$\dot{X} = \begin{pmatrix} [a_{11}] & [a_{12}] & -9.8 \\ [a_{21}] & [a_{22}] & 0 \\ 0 & 1 & 0 \end{pmatrix} X + \begin{pmatrix} [b_{11}] & 0 \\ 0 & [b_{22}] \\ 0 & 0 \end{pmatrix} u \tag{16}$$

with $[a_{11}] = [-0.031, -0.0128]$, $[a_{12}] = [-3.4, -0.1]$, $[a_{21}] = [-0.00077, 0.0007]$, $[a_{22}] = [-0.32, -0.31]$, $[b_{11}] = [-18, -15]$ and $[b_{22}] = [-3.3, -3]$.

Using the same notation in (11) with a state feedback gain matrix $K = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{pmatrix}$, the state matrix A_c of the closed loop system is

$$[A_c] = \begin{pmatrix} [a_{11}]-k_{11}[b_{11}] & [a_{12}]-k_{12}[b_{11}] & -9.8-k_{13}[b_{11}] \\ [a_{21}]-k_{21}[b_{22}] & [a_{22}]-k_{22}[b_{22}] & -k_{23}[b_{22}] \\ 0 & 1 & 0 \end{pmatrix} \tag{17}$$

and its second additive compound matrix is given by:

$$[A_c]^{[2]} = \begin{pmatrix} \begin{matrix}[a_{11}]-k_{11}[b_{11}] \\ +[a_{22}]-k_{22}[b_{22}]\end{matrix} & -k_{23}[b_{22}] & 9.8+k_{13}[b_{11}] \\ 1 & [a_{11}]-k_{11}[b_{11}] & [a_{12}]-k_{12}[b_{11}] \\ 0 & [a_{21}]-k_{21}[b_{22}] & [a_{22}]-k_{22}[b_{22}] \end{pmatrix} \tag{18}$$

Applying proposition 2, a sufficient stability condition of system (16) is given by the following inequalities:

$$\begin{cases} \det([A_c]) = (k_{21}[b_{22}][b_{11}]-[a_{21}][b_{11}])k_{13} \\ \qquad +([b_{22}][a_{11}]-[b_{22}][b_{11}]k_{11})k_{23} \\ \qquad +9.8[b_{22}]k_{21}-9.8[a_{21}]<0 \\ [a_{11}]-k_{11}[b_{11}]+[a_{22}]-k_{22}[b_{22}]+1<0 \\ \left|k_{23}[b_{22}]\right| + [a_{11}]-k_{11}[b_{11}] + \left|[a_{21}]-k_{21}[b_{22}]\right| < 0 \\ \left|9.8+k_{13}[b_{11}]\right| + \left|[a_{12}]-k_{12}[b_{11}]\right| + [a_{22}]-k_{22}[b_{22}] < 0 \end{cases} \tag{19}$$

According to the system of inequalities above, we can look a priori for a state feedback control law involving only entries k_{11}, k_{22} and k_{23} in the gain matrix K, i.e. $k_{12} = k_{13} = k_{21} = 0$. Moreover, given the signs of interval parameters weighted with the nonzero entries of the gain matrix, it comes:

$$\begin{cases} k_{11},\ k_{22} < 0 \\ k_{21} > 0 \end{cases} \tag{20}$$

By applying basic interval arithmetic operations on system (19), it reduces to

$$\begin{cases} [-32.34, -29.4]k_{21} + [-0.686\,10^{-2}, 0.7546\,10^{-2}] < 0 \\ [-0.351, -0.3228]-[-18, -15]k_{11}-[-3.3, -3]k_{22}+1 < 0 \\ [-0.77\,10^{-3}, 0.7\,10^{-3}]-[-3.3, -3]k_{21} < [0.031, 0.0128] \\ \qquad\qquad +[-18, -15]k_{11} \\ [-0.031, -0.0128]-[-18, -15]k_{11} < [-0.77\,10^{-3}, 0.7\,10^{-3}] \\ \qquad\qquad -[-3.3, -3]k_{21} \\ 9.8+[-0.22, 3.09]-[-3.3, -3]k_{22} < 0 \end{cases} \tag{21}$$

Sufficient conditions for the above inequalities to be hold are obtained using majorization and can easily be solved:

$$\begin{cases} -29.4k_{21}+0.7546\,10\text{-}2 < 0 \\ 0.6772+15k_{11}+3k_{22} < 0 \\ -0.0121+3.3k_{21}+15k_{11} < 0 \\ -0.01203+15k_{11}-3k_{21} < 0 \\ 12.89+3k_{22} < 0 \end{cases} \tag{22}$$

Solving the system (22) provides a choice for the nonzero gain matrix entries:

$$\begin{cases} k_{11} < 0.75020\,10^{-3} \\ k_{21} > 0.25666\,10^{-3} \\ -0.01210+3.3k_{21}+15k_{11} < 0 \\ k_{22} < -4.29666 \end{cases} \tag{23}$$

One possible choice is $\{k_{11} = -0.3,\ k_{21} = 2,\ k_{22} = -5\}$.

The corresponding interval characteristic polynomial $P(\lambda)$ for the matrix A_c can be deduced using interval arithmetic and is given by:

$$[P(\lambda)] = \lambda^3 + [19.8228, 22.251]\lambda^2 + [69.691038, 113.786802]\lambda + [58.80686, 64.672454] \tag{24}$$

The interval polynomial above is Hurwitz stable according to the eigenvalues of the four corresponding Kharitonov polynomials listed below.

$$\begin{matrix} \{-18.69, -1.93, -1.62\} \\ \{-14.86, -6.81, -0.58\} \\ \{-9.59-3.09i, -9.59+3.09i, -0.63\} \\ \{-15.63, -2.61, -1.58\} \end{matrix} \tag{25}$$

The stability of the controlled interval system is confirmed by means of numerical simulation. Figure 1 illustrates the evolution of the system states for different values of the uncertain parameters.

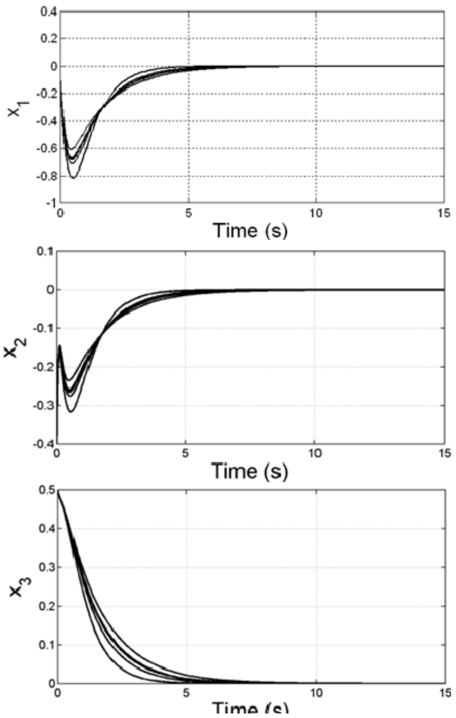

Fig. 1. Evolution of the controlled system states for different values of the uncertain parameters

V. CONCLUSION

The compound matrix approach is adopted to derive robust stability conditions for interval systems. Obtained conditions are used for robust state feedback stabilization. The stabilizing controller parameters are solutions of a linear inequality system and only one nonlinear inequality. Solving such inequalities can be performed easily using symbolic calculation software. The results obtained in this paper propose an alternative to those derived using the Kharitonov theory for interval linear systems, but can be extended to the nonlinear context. Numerical simulations are presented to illustrate the application of the proposed approach.

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