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Optimal Control Design For A Class Of Discrete-Time Systems

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Abstract — **In this paper, we are interested in the optimal control of discrete-time complex systems. The multi-model approach was proposed as a solution and a powerful tool for the description, the control and the analysis of the studied systems. Minimizing a global quadratic performance index, necessary conditions of existence of the optimal control gain matrices are proposed and an iterative algorithm is presented to compute all the introduced results. By the Lyapunov stability theory, we proved that these optimal gains ensure the asymptotic stability of the controlled system. To demonstrate the effectiveness of the proposed strategy simulation results are given.**

Keywords—optimal control; output feedback control; nonlinear systems; discrete-time systems

I. INTRODUCTION

The control of nonlinear systems have been the subject of numerous research papers [1-4,7,10-14,20]. However, the static and dynamic output feedback control of discrete-time nonlinear systems has received little attention. When the state vector of the considered system is not completely available for feedback, output feedback solutions can be considered. The static output feedback problem is known to be a challenging issue due to its non-convex nature and many attempts have been made to solve the linear case problem [5-16]. The nonlinear case continue to attract the attention of many researchers [15-18].

In practice an accurate model of the studied plant can't be usually available, might change and can have different operating points. To overcome these difficulties, the multi-model approach [9-13] was proposed in the literature as a solution and a powerful tool for the description, the control and the analysis of the studied systems. Therefore, motivated by this result, we investigate the optimal control problem for nonlinear uncertain systems described by a discrete-time multi-model representation. The optimization problem was reduced to the minimization of a global quadratic performance index having a direct signification and interpretation regarding to the convergence and the control law of the controlled system. Necessary conditions of existence of the optimal control gain matrices are proposed.

The rest of the paper is organized as follows: In Section II, the description of the studied systems and problem formulation is given. Section III gives the main results of this work. In section IV, an example is given to illustrate the proposed method. Section V contains the conclusions and final remarks.

II. PROBLEM STATEMENTS

Consider in this study a class of nonlinear and uncertain discrete-time system described by:

$$(S)\begin{cases} x(k+1) = f\left(x(k), u(k), \theta(k)\right) \\ y(k) = h\left(x(k), \theta(k)\right) \\ \theta(k) \in ID \end{cases} \tag{1}$$

where $x(k) \in \Re^n$ is the state vector, $u(k) \in \Re^m$ is the input vector and $y(k) \in \Re^p$ is the output vector. The functions $f(.)$ and $h(.)$ depend on a vector of parameters $\theta(k)$ which is considered unknown, but evolving in a convex domain ID.

In the literature, various approaches [13] like identification, linearization or convex polytopic transformation can be used to determine the multimodel description of a complex system.

In this paper, we assume that the nonlinear mathematical model of the studied system is known . By linearization around its several operating points $(u_{i0}, x_{i0}), i = 1...N$, different and simpler local models are obtained. So the complex studied system described initially by a nonlinear mathematical model (1) can be then described by a library of N local linear model characterized by the following state space equations:

$$(M_i)\begin{cases} x_i(k+1) = A_i x_i(k) + B_i u_i(k) \\ y_i(k) = C_i x_i(k) \\ i = 1, \dots, N \end{cases} \tag{2}$$

where N *is* the number of local models, $x_i(k) \in \Re^n$,

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$y_i(k) \in \Re^p$ and $u_i(k)$ are respectively the state vector, the output vector and the control input vector of the *i-th* submodel noted M_i.

The state space matrices A_i, B_i, C_i are constant of appropriate dimensions to be determined.

$$A_i = \frac{\partial f}{\partial x}\bigg|_{(u_{i0}, x_{i0})}, B_i = \frac{\partial f}{\partial u}\bigg|_{(u_{i0}, x_{i0})}, C_i = \frac{\partial h}{\partial x}\bigg|_{(u_{i0}, x_{i0})} \quad .(3)$$

and let's note:

$$\begin{cases} x_i(k) = x(k) - x_{i0} \\ u_i(k) = u(k) - u_{i0} \\ i = 1, \ldots, N \end{cases} \quad (4)$$

Each model of the library, involving N sub-models, contributes to the process description with a degree of trust measured by a validity coefficient. The validity appears to be of great importance if realizing their influence on the performances of the global control law [9,13].

Indeed, the use of the validity coefficients is a convenient mean to experiment with sub-collection of systems and is also useful to put more emphasis on the performances of some particular instances of parameter values. In the literature several methods were proposed for the estimation of these validities. In this paper the residue approach [13] is considered for validities computing.

III. Optimal output feedback control for discrete-time nonlinear system

In this section, our objective is to design an output feedback controller for the studied system and a cost function such that the resulting closed-loop system is asymptotically stable and the closed loop cost function is minimized.

Assume for each isolated subsystem M_i, a local controller is designed.

$$u_i(k) = -F_i y_i(k) \quad (5)$$

where $F_i \in \Re^{m \times p}$ is the control gain matrix to be determined by minimizing the proposed quadratic function:

$$J_i = \sum_{k=0}^{\infty} \left(x_i^T(k) Q_i x_i(k) + u_i^T(k) R_i u_i(k) \right) \quad (6)$$

where $Q_i = Q_i^T \geq 0 \in \Re^{n \times n}$ and $R_i = R_i^T > 0 \in \Re^{m \times m}$ are the state and input weighting matrices.

Applying controller (5) to the system (2) results in the closed-loop system:

$$x_i(k+1) = (A_i - B_i F_i C_i) x_i(k) \quad (7)$$

The performance index associated with the studied system (1) is then the following quadratic function:

$$\begin{aligned} J &= \sum_{i=1}^{N} \mu_i J_i \\ &= \sum_{i=1}^{N} \mu_i \sum_{k=0}^{\infty} x_i^T(k)(Q_i + C_i^T F_i^T R_i F_i C_i) x_i(k) \end{aligned} \quad (8)$$

where μ_i, $i = 1, \ldots, N$ are the validity coefficients of the proposed multimodel description

Using the solution of the recurrent equation (7), one can write:

$$x_i(k) = (A_i - B_i F_i C_i)^K x_i(0) \quad (9)$$

and substituting (9) in (8), the global performance index (8) can be rewritten :

$$J = \sum_{i=1}^{N} \mu_i \sum_{k=0}^{\infty} x_i^T(0) \left[\left((A_i - B_i F_i C_i)^K \right)^T \left(Q_i + C_i^T F_i^T R_i F_i C_i \right) (A_i - B_i F_i C_i)^K \right] x_i(10)$$

and presented in a simplified expression :

$$J = \sum_{i=0}^{N} \mu_i x_{i0}^T P_i x_{i0} \quad (11)$$

where

$$P_i = \sum_{k=0}^{\infty} \left[\left((A_i - B_i F_i C_i)^K \right)^T \left(Q_i + C_i^T F_i^T R_i F_i C_i \right) (A_i - B_i F_i C_i)^K \right]$$

are symmetric positive definite matrices, solutions of the following Lyapunov equations:

$$\left(A_i - B_i F_i C_i \right)^T P_i \left(A_i - B_i F_i C_i \right) - P_i + Q_i + C_i^T F_i^T R_i F_i C_i = 0 \; ; \quad i = 1, ..., N \quad (12)$$

The dependency of the optimal solution on the initial condition can be removed when considering the average value function $E(.)$ such that:

$$E\left(x_{i0}^T P_i x_{i0} \right) = I_n \quad (13)$$

Based on equation (13), the corresponding closed-loop cost function will be written as follows :

$$\bar{J} = \sum_{i=1}^{N} \mu_i trace\{P_i\} \quad (14)$$

A- Main results

In order to derive the necessary conditions of optimal gain matrices of the feedback control, the optimization problem formulated by (11) is reduced to the minimization of the following Lagrangian:

$$\begin{aligned} \zeta(F_i, P_i, S_i) = & \sum_{i=1}^{N} \mu_i \, trace\{P_i\} + \\ & \sum_{i=1}^{N} \mu_i \, trace\Big\{ \Gamma_i^T \Big[\left((A_i - B_i F_i C_i)^T P_i (A_i - B_i F_i C_i) \right. \\ & \left. - P_i + Q_i + C_i^T F_i^T R_i F_i C_i \right) \Big] \Big\} \end{aligned} \quad (15)$$

where $\Gamma_i \in \Re^{n\times n}, \Gamma_i = \Gamma_i^T \geq 0, i = 1,...,N$ selected to be symmetric positive definite matrices are Lagrange multipliers.

By using the gradient matrix operations [8], the necessary conditions for F_i, P_i and Γ_i, to be optimal are given by

$$\begin{cases} \frac{\partial\zeta(F_i,P_i,\Gamma_i)}{\partial F_i} = -2\sum_{i=1}^{N}\mu_i\left[B_i^T P_i A_i \Gamma_i C_i^T - B_i^T P_i B_i F_i C_i \Gamma_i C_i^T - R_i F_i C_i \Gamma_i C_i^T\right] = 0 \\ \frac{\partial\zeta(F_i,P_i,\Gamma_i)}{\partial P_i} = \sum_{i=1}^{N}\mu_i\left[(A_i - B_i F_i C_i)\Gamma_i (A_i - B_i F_i C_i)^T - \Gamma_i + I_n\right] = 0 \\ \frac{\partial\zeta(F_i,P_i,\Gamma_i)}{\partial \Gamma_i} = \sum_{i=1}^{N}\mu_i\left[(A_i - B_i F_i C_i)^T P_i (A_i - B_i F_i C_i) - P_i + Q_i + C_i^T F_i^T R_i F C_i\right] = 0 \\ i=1,...,N \end{cases} \quad (16)$$

solving the first equation in (16), one obtains the optimal control gain matrix F_i of the local model M_i:

$$F_i = \left(B_i^T P_i B_i + R_i\right)^{-1}\left(B_i^T P_i A_i \Gamma_i C_i^T\right)\left(C_i \Gamma_i C_i^T\right)^{-1} \quad (17)$$

and from the two others equations we can determine the matrices Γ_i and all the matrices P_i solutions of the Lyapunov equations (12). Indeed, based on (16), Γ_i and P_i are also the solutions of the following equations:

$$\begin{cases} G_2(F_i,\Gamma_i) = 0 \\ G_3(F_i,P_i) = 0 \end{cases}$$

$$G_2(F_i,\Gamma_i) = (A_i - B_i F_i C_i)\Gamma_i (A_i - B_i F_i C_i)^T - \Gamma_i + I_n \quad (18)$$

$$G_3(F_i,P_i) = (A_i - B_i F_i C_i)^T P_i (A_i - B_i F_i C_i) - P_i + Q_i + C_i^T F_i^T R_i F_i C_i \quad (19)$$

To solve instantly the three equations (17), (18) and (19), and to calculate all the introduced matrices F_i, P_i and Γ_i, we propose an iterative algorithm which can be summarized in the following way:

Algorithm

Step1 : Initialize $n = 1$

Select $Q_i \geq 0, R_i > 0$ and an initial matrix F_{i0} u as initial starting value such that $A_i - B_i F_{i0} C_i$ is a stable for each local model.

Step 2 : n[th] iteration

- calculate F_{in} (17)
- solve $G_2(F_{in},P_{in}) = 0$ and calculate Γ_{in}.
- solve $G_3(F_{in},P_{in}) = 0$; and get the matrix P_{in}.
- calculate

$$F_{i(n+1)} = \left(B_i^T P_{in} B_i + R_i\right)^{-1}\left(B_i^T P_{in} A_i \Gamma_{in} C_i^T\right)\times\left(C_i \Gamma_{in} C_i^T\right)^{-1}$$

Step 3 : incrementation

repeat step 2 until verifying $\left\|P_{in} - P_{i(n-1)}\right\| \leq \varepsilon$

End

ε is a prescribed small number used to check the convergence of the algorithm

B- The optimal controller design

Given the predetermined matrices F_i the system (1) can be controlled in an optimal manner by the following control policy $u(k)$ which guarantees the minimization of the infinite horizon cost function (8).

$$u(k) = -\sum_{j=1}^{N}\mu_j F_j\ y(k) \quad (20)$$

then the closed-loop system (1) admits the realization:

$$\begin{cases} x(k+1) = f\left(x(k),u(k),\theta(k)\right) \\ y(k) = h\left(x(k),u(k)\right) \\ u(k) = -Fy(k) \end{cases}$$

where

$$F = \sum_{i=1}^{N}\mu_i F_i \ , \ F \in \Re^{p\times m} \quad (21)$$

C - Stability analysis

In order to prove the asymptotic stability of the controlled system, let's consider $V(x_i(k))$ the Lyapunov function defined by the following quadratic form:

$$V(x_i(k)) = x_i^T(k) P_i x_i(k) \quad (22)$$

where $P_i \in \Re^{n\times n}$ are the symmetric positive definite matrices solution of the equation (12) and (16).

The stability of the controlled system (22) is ensured if the difference of Lyapunov function (18) along the trajectory of (7) is negative definite.

One has:

$$\Delta V(k) = V(x_i(k+1)) - V(x_i(k))$$
$$= x_i^T(k+1)P_i x_i(k+1) - x_i^T(k)P_i x_i(k) \quad (23)$$
$$= x_i^T(k)\left[(A_i - B_i F_i C_i)^T P_i (A_i - B_i F_i C_i) - P_i\right]x_i(k)$$

Using the third equation of system (16), (19) becomes:

$$\Delta V(k) = -x_i^T(k)\left[Q + C_i^T F_i^T R_i F_i C_i\right]x_i(k) \quad (24)$$

According to the properties of matrices Q_i and R_i, the matrix $Q_i + C_i^T F_i^T R F_i C$ is symmetric positive definite. The variation of the quadratic Lyapunov function, expressed by (24), is then negative defined and the controlled system is then asymptotically stable.

IV. NUMERICAL EXAMPLE

Let's consider the mechanical system described by a spring damper mass M [25] as follows:

$$M\ddot{x}(t) + c_1\dot{x}(t) + c_2 x(t) = \left(1 + c_3\dot{x}^3(t)\right)u(t) \quad (25)$$

where:

- $M = 1Kg$ is the mass of the system,
- $c_1 = 1$, $c_2 = 1.155$ and $c_3 = 0.13$ are constants,
- $u(t)$ is the exerted force for the spring,
- $\dot{x}^3(t)$ is the nonlinear term.

and rewritten in the following state space equations:

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1}{M} & -\frac{c_2}{M} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{M}\left(1 + c_3 x_1^3(t)\right) \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} x_1(t) \\ 2x_1(t) + x_2(t) \end{bmatrix} \end{cases} \quad (26)$$

where $x_1(t)$ is the velocity of the mass and $x_2(t)$ the posit ion of the same mass.

By using an appropriate discretization method and a suitable sampling period $T = 0.05s$, it comes the discrete-time state space equations:

$$\begin{cases} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 - T\frac{c_1}{M} & -T\frac{c_2}{M} \\ T & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{T}{M}\left(1 + c_3 x_1^3(k)\right) \\ 0 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{cases}$$

we pointed out that the nonlinearity of the system is considered as uncertainty and the term of linearities depend on $x_1(k)$ which is assumed to vary in the range $[-1.5 \quad 1.5]$.

According to section III, and based on the multimodel approach the nonlinear dynamical system (26) can be described by:

$$\begin{cases} x(k+1) = \sum_{i=1}^{2} \mu_i(x_1(k))\left(A_i x(k) + B_i u(k)\right) \\ y(k) = \sum_{i=1}^{2} \mu_i(x_1(k)) C_i x(k) \end{cases} \quad (27)$$

where $x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$ and $A_1 = A_2 = \begin{bmatrix} 0.9 & -0.1155 \\ 0.1 & 1 \end{bmatrix}$,

$B_1 = \begin{bmatrix} 0.1439 \\ 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0.0561 \\ 0 \end{bmatrix}$, $C_1 = C_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

The validity coefficients of this system are expressed as follows:

$$\mu_1(x_1(k)) = 0.5 + \frac{x_1^3(k)}{6.75}$$
$$\mu_2(x_1(k)) = 1 - \mu_1(x_1(k))$$

Using the proposed iterative algorithm the following results are derived:

- The quadratic criterion:
  $J_1 = 0.7727$, $J_2 = 0.9708$
- The symmetric positive definite matrices:
  $P_1 = \begin{bmatrix} 0.1412 & 0.1148 \\ 0.1148 & 0.6315 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0.2735 & 0.1806 \\ 0.1806 & 0.6973 \end{bmatrix}$
- The symmetric positive definite matrices of Lagrange multipliers:
  $\Gamma_1 = \begin{bmatrix} 5.9552 & -5.2978 \\ -5.2978 & 7.0525 \end{bmatrix}$, $\Gamma_2 = \begin{bmatrix} 6.9638 & -5.3482 \\ -5.3482 & 9.8512 \end{bmatrix}$

and all the gain matrices of the proposed optimal control are calculated:

$$F_1 = [-1.0621 \quad 1.7891], \quad F_2 = [-0.3236 \quad 1.4268]$$

To show the effectiveness of the proposed optimal output feedback control we have carried out some simulations shown from figure 1 to 3. It appears a satisfactory stabilization of the state variables of the controlled discrete-time studied system (figures 1 to 2). The figure 3 illustrates the evolution of the proposed optimal output feedback control law. Indeed, its high performances shows the aptitude of the proposed

Algorithm to be implemented and to give interesting results for the output feedback control of a large class of nonlinear discrete-time systems.

V. Conclusion

The nonlinear discrete-time studied system is first represented by a multi local linear models. Then, an output feedback controller based on the multimodel control approach and minimizing a quadratic criterion is derived assuring the asymptotic stability of the controlled system. An illustrative example is considered and the simulation results show the effectiveness of the proposed control strategy.

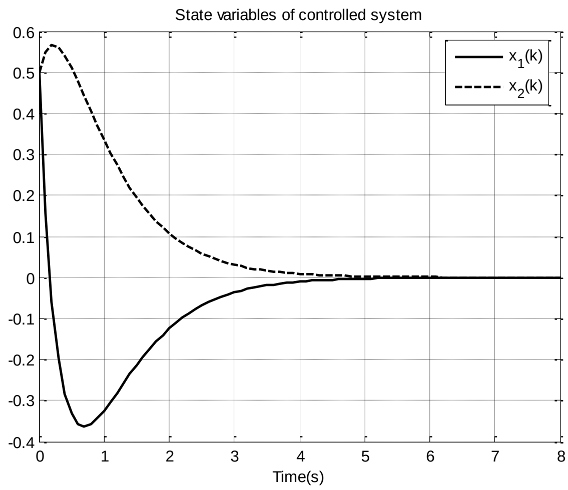

Figure 1: Evolution of the state vector of the system provided with the optimal proposed control law

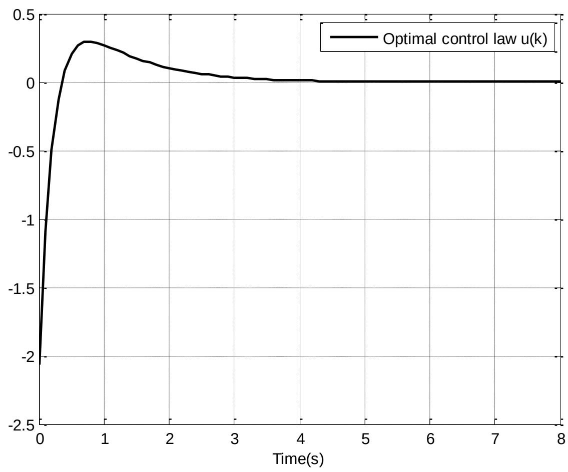

Figure 2: Evolution of the optimal multimodel control law

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