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NARMAX structure and identification of coupled mass-spring-damper system

Sana RANNEN
Laboratory of Advanced Systems,
Polytechnic High School of Tunisia,
LSA - EPT, University of Carthage,
BP 743, 2078 La Marsa
sana.rannen@gmail.com

Chekib GHORBEL
Laboratory of Advanced Systems,
Polytechnic High School of Tunisia,
LSA - EPT, University of Carthage,
BP 743, 2078 La Marsa
chkb.ghorbel@gmail.com

Naceur BENHADJ BRAIEK
Laboratory of Advanced Systems,
Polytechnic High School of Tunisia,
LSA - EPT, University of Carthage,
BP 743, 2078 La Marsa
naceurbhb@gmail.com

***Abstract*—Most physical processes encountered in the real world are nonlinear in nature. The NARMAX (Nonlinear AutoRegressive Moving Average eXogenous inputs) model provides a unified representation for a wide class of nonlinear system. The problem considered in this work is the parametric estimation of coupled mass-spring-damper system described by nonlinear structures.**

I. Introduction

Mathematical models are an essential part of most branches of science and engineering. Indeed, a mathematical model can be used to unveil fundamental properties of a system which are not apparent otherwise, leading to a better understanding of that process, or in the design of an automatic control that can be used to regulate the behavior of certain system variables.

This expression can be obtained directly from experimental input/output data by determining the model form and the numerical values of the unknown parameters or it can be established using analytical development and physical laws, which is difficult in most cases. To deal with this problem, the process behaviour can be approximated using the system identification theory that aims to determinate mathematical model of a system with sufficient accuracy [1].

Most systems encountered in the real world are nonlinear in nature, and since linear models cannot capture the rich dynamic behavior of limit cycles, bifurcations, etc. associated with nonlinear systems, it is imperative to have identification techniques which are specific for nonlinear systems [2].

Leontaritis and Billings, [3][4], have proposed the Nonlinear AutoRegressive Moving Average eXogenous inputs (NARMAX) structure as a general parametric form for modeling nonlinear systems structure. NARMAX model is capable of describing a wide variety of nonlinear systems. This formulation yields compact model descriptions that may be readily identified and may afford greater interpretability [5].

This paper aims to model and identify nonlinear coupled mass-spring-damper system using NARX and NARMAX structures.

II. NARMAX model

NARMAX model provides a unified representation for a wide class of nonlinear system. NARMAX structures models the input-output relationship as a nonlinear difference equation of the form [5] [6]:

$$\begin{aligned} y(k) = f\,(& y(k-1), y(k-2), \cdots, y(k-n_y), \\ & u(k-1), u(k-2), \cdots, u(k-n_u), \\ & e(k-1), e(k-2), \cdots, e(k-n_e)) \end{aligned} \quad (1)$$

where $f(.)$ is a nonlinear function, $u(k)$, $y(k)$ and $e(k)$ are input, output and noise signal respectively and n_u, n_y and n_e are their associated maximum lags. This nonlinear mapping may include a variety of nonlinear terms, such as terms raised to an integer power $\left[\text{e.g.}, u^2(k-3)\right]$, products of present and past inputs $\left[\text{e.g.}, u(k)\,u(k-1)\right]$, past outputs $\left[\text{e.g.}, y(k-1)\,y(k-2)\right]$ or cross-terms $\left[\text{e.g.}, u^2(k-1)\,y(k-2)\right]$. This system description encompasses most forms of nonlinear difference equations that are linear in the parameters.

A special case of the general NARMAX model is the NARX (Nonlinear AutoRegressive with eXogenous inputs) model [7]:

$$\begin{aligned} y(k) = f\,(& y(k-1), y(k-2), \cdots, y(k-n_y), \\ & u(k-1), u(k-2), \cdots, u(k-n_u)) \end{aligned} \quad (2)$$

III. Modelisation of coupled mass-spring-damper system

A model of coupled mass-spring-damper system (two degree of freedom system) is shown in Fig. 1 [8].
The model is composed of two nonlinear springs, two weights and two dampers. Since the upper mass is attached to both springs, there are two nonlinear springs restoring forces (Hooke's law) acting upon it: an upward force f_{r1} exerted by the elongation (or compression) x_1 of the first spring; an upward force f_{r2} from the second spring resistance to being

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elongated (or compressed) by the amount $x_2 - x_1$.

The second mass only feels the nonlinear restoring force from the elongation (or compression) of the second spring. Allowing the system to come and to rest in equilibrium, we measure the displacement of the centre of mass of each weight from equilibrium, as a function of time, and denote these measurement by $x_1(t)$ and $x_2(t)$ respectively.

where:
$f_{r1} = -k_1x_1 + \mu_1 x_1^3 - k_2(x_1 - x_2) + \mu_2(x_1 - x_2)^3$ and $f_{r2} = -k_2(x_2 - x_1) + \mu_2(x_2 - x_1)^3$,
k_i: spring constant, with $i = 1, 2$
x_i: displacement of the centre of mass of the moving object,
m_i: mass of the moving object,
δ_i: damping coefficient,
μ_i: nonlinear coefficient.

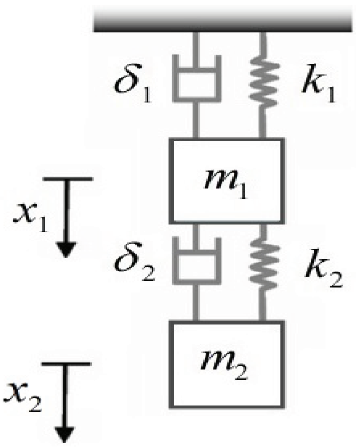

Fig. 1 – Coupled mass-spring-damper

Newton's Law implies that two equations representing the motions of the weights are:

$$\left\{\begin{array}{l} m_1\ddot{x}_1(t) = -\delta_1\dot{x}_1(t) - k_1x_1(t) + \mu_1x_1^3(t) \\ \quad -k_2(x_1(t) - x_2(t)) \\ \quad +\mu_2(x_1(t) - x_2(t))^3 + u_1(t) \\ m_2\ddot{x}_2(t) = -\delta_2\dot{x}_2(t) - k_2(x_2(t) - x_1(t)) \\ \quad +\mu_2(x_2(t) - x_1(t))^3 + u_2(t) \end{array}\right. \tag{3}$$

According to the pair of coupled second-order nonlinear differential equations (3), we have:
$\mu_2(x_1(t) - x_2(t))^3 = -\mu_2(x_2(t) - x_1(t))^3$,

then:

$$\begin{array}{l} m_1\ddot{x}_1(t) + \delta_1\dot{x}_1(t) + k_1x_1(t) - \mu_1x_1^3(t) \\ +k_2(x_1(t) - x_2(t)) - u_1(t) = -\delta_2\dot{x}_2(t) \\ -k_2(x_2(t) - x_1(t)) - m_2\ddot{x}_2(t) + u_2(t) \end{array} \tag{4}$$

Using the finite difference approximation the coupled mass-spring-damper model given in continuous time can be converted to discrete time.

A. Finite difference approximation

Finite difference techniques rely on the approximation of a derivative as the difference in the dependent variable over a small interval of the independent variable, those approximations are written using a small set of difference operators [9]. Then, approximation of derivative operator is an approach which consists in approximating a derivative operator using a linear combination of signal values [10].

Taylor series is among the techniques used in the approximation of derivative operator.

$$\begin{array}{l} f(x_{k+1}) \approx f(x_k) + \frac{df}{dx}(x_k)\,T \\ + \frac{d^2f}{dx^2}(x_k)\,\frac{T^2}{2!} + \ldots + \frac{d^nf}{dx^n}(x_k)\,\frac{T^n}{n!} \end{array} \tag{5}$$

where $T = x_{k+1} - x_k$ is the interval over which we wish to approximate the derivative and the superscripts indicate the order of the derivative of f(x). Truncating after the first derivative and rearranging yields an approximation of the first derivative:

$$\frac{df}{dx}(x_k) \approx \frac{f(x_{k+1}) - f(x_k)}{T} \tag{6}$$

For the second derivative:

$$\frac{d^2f}{dx^2}(x_k) \approx \frac{f(x_{k+2}) - 2f(x_{k+1}) + f(x_k)}{T^2} \tag{7}$$

B. Discrete model of coupled mass-spring-damper

Applying the approximation of derivative operator approach equation (4) can be converted to discrete time and rewritten as a NARX models (9) and (10).

$$\begin{array}{l} \frac{m_1}{T^2}x_1(k+2) + \left(\frac{\delta_1}{T} - \frac{2m_1}{T^2}\right)x_1(k+1) \\ + \left(\frac{m_1}{T^2} - \frac{\delta_1}{T} + k_1\right)x_1(k) - \mu_1 {x_1}^3(k) \\ + \frac{m_2}{T^2}x_2(k+2) + \left(\frac{\delta_2}{T} - \frac{2m_2}{T^2}\right)x_2(k+1) \\ + \left(\frac{m_2}{T^2} - \frac{\delta_2}{T}\right)x_2(k) = u_1(k) + u_2(k) \end{array} \tag{8}$$

Equation (8) can be described by:

$$\begin{array}{l} x_1(k) = -\ \alpha_{11}\ x_1(k-1) - \alpha_{12}\ x_1(k-2) \\ \quad + \alpha_{13}\ {x^3}_1(k-2) - \beta_{11}\ x_2(k) \\ \quad - \beta_{12}\ x_2(k-1) - \beta_{13}\ x_2(k-2) \\ \quad + \gamma_{11}\ u_1(k-2) + \gamma_{12}\ u_2(k-2) \end{array} \tag{9}$$

TABLE I – Theoretical relationship of NARX parameters to continuous-time system coefficients

NARX coefficient	Relationship to continuous-time coefficient
α_{11}	$(\frac{\delta_1}{T} - \frac{2m_1}{T^2})\frac{T^2}{m_1}$
α_{12}	$(\frac{m_1}{T^2} - \frac{\delta_1}{T} + k_1)\frac{T^2}{m_1}$
α_{13}	$\mu_1\frac{T^2}{m_1}$
β_{11}	$\frac{m_2}{T^2}\frac{T^2}{m_1}$
β_{12}	$(\frac{\delta_2}{T} - \frac{2m_2}{T^2})\frac{T^2}{m_1}$
β_{13}	$(\frac{m_2}{T^2} - \frac{\delta_2}{T})\frac{T^2}{m_1}$
$\gamma_{11} = \gamma_{12}$	$\frac{T^2}{m_1}$

$$x_2(k) = -\alpha_{21}\, x_2(k-1) - \alpha_{22}\, x_2(k-2) - \beta_{21}\, x_1(k) - \beta_{22}\, x_1(k-1) - \beta_{23}\, x_1(k-2) + \beta_{24}\, {x_1}^3(k-2) + \gamma_{21}\, u_1(k-2) + \gamma_{22}\, u_2(k-2) \tag{10}$$

TABLE II – Theoretical relationship of NARX parameters to continuous-time system coefficients

NARX coefficient	Relationship to continuous-time coefficient
α_{21}	$(\frac{\delta_2}{T} - \frac{2m_2}{T^2})\frac{T^2}{m_2}$
α_{22}	$(\frac{m_2}{T^2} - \frac{\delta_2}{T})\frac{T^2}{m_2}$
β_{21}	$\frac{m_1}{T^2}\frac{T^2}{m_2}$
β_{22}	$(\frac{\delta_1}{T} - \frac{2m_1}{T^2})\frac{T^2}{m_2}$
β_{23}	$(\frac{m_1}{T^2} - \frac{\delta_1}{T} - k_1)\frac{T^2}{m_2}$
β_{24}	$\mu_1\frac{T^2}{m_2}$
$\gamma_{21} = \gamma_{22}$	$\frac{T^2}{m_1}$

Table 1 and 2 show the relationship of discrete-time NARX parameters in (9) and (10) to the underlying continuous-time coefficients.

IV. Identification of NARX model of coupled mass-spring-damper

There are several algorithms of identification and the most used is this called RLS (Recursive Least Square algorithm). Indeed, it is frequently used because of his capability to approximate a large class of systems and his simplicity of implementation [11].

A. Identification with RLS algorithm

As shown in Fig. 2, RLS algorithm allows to estimate the model parameters by minimizing a measure of the model prediction error

$$\varepsilon(k) = y(k) - \hat{y}(k) \tag{11}$$

where $\hat{y}(k)$ is the prediction of the scalar measured output $y(k)$, given by:

$$\hat{y}(k) = \hat{\theta}^T(k-1)\,\psi(k) \tag{12}$$

where $\hat{\theta}(k)$ is the vector of estimated parameters and $\psi(k)$ is the regression vector containing old inputs and outputs of the system to be identified [12].

The RLS algorithm can be written in following form:

$$\begin{cases} \hat{\theta}(k) = \hat{\theta}(k-1) + P(k)\,\psi(k)\,\varepsilon(k) \\ P(k) = P(k-1) - \frac{P(k-1)\psi(k)\psi^T(k)P(k-1)}{1+\psi^T(k)P(k-1)\psi(k)} \\ \varepsilon(k) = y(k) - \hat{y}(k) \end{cases} \tag{13}$$

with $P(k)$ is the gain matrix, given by:

$$P(k) = \left[\sum_{i=n+1}^{k} \psi(i)\,\psi^T(i)\right]^{-1} \tag{14}$$

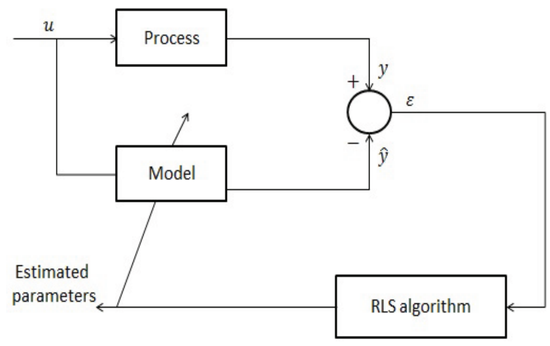

Fig. 2 – RLS identification method

B. Identification of NARX model

Fig. 3 shows the simulation input $x_2(k)$ and predicted output of the NARX description model.

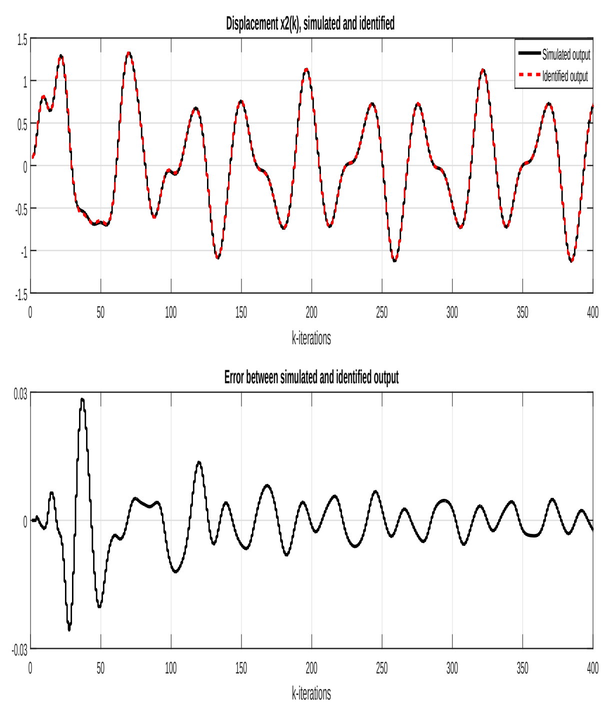

Fig. 3 – (top) Displacement $x_2(k)$, simulated and identified. (bottom) Error betwen simulated and identified output

Fig. 4 presents the results of identifying the simulated model. This figure shows that the identified parameters values corresponded closely to those derived theoretically.

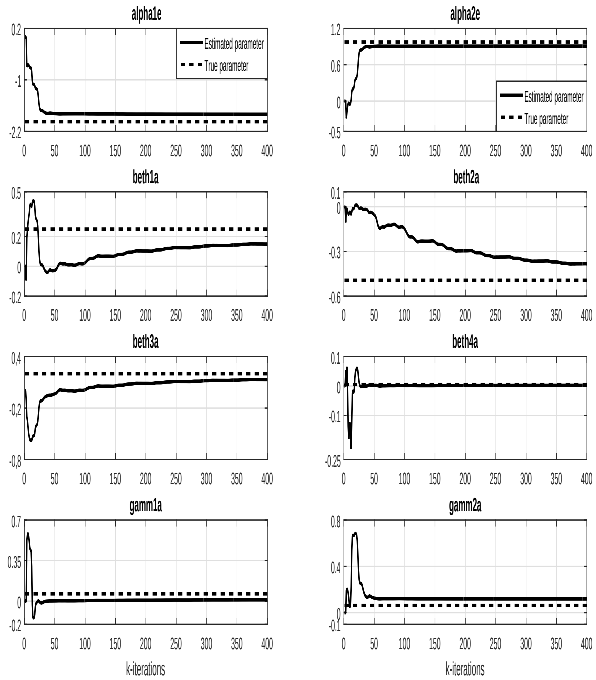

Fig. 4 – True and estimated parameters

C. Model validation

It is not obvious that how to quantify the analysis of the model simulated data with the measurements. In order to solve this problem, the equation (15) is used to evaluate the accuracy of model output.

$$\%\text{VAF} = \left(1 - \frac{\frac{1}{N}\sum_{k=1}^{N}\left(y\left(k\right) - \hat{y}\left(k\right)\right)^2}{\frac{1}{N}\sum_{k=1}^{N}\left(y\left(k\right)\right)^2}\right) \tag{15}$$

where N is the record length, $y\left(k\right)$ is the measured data, $\hat{y}\left(k\right)$ is the model output. The simulation output $\hat{y}\left(k\right)$ of the NARX description model is compared with the output of the measured data $y\left(k\right)$ by computing the percent variance for %VAF.

To determine the validity of this NARX description model (10) we simulated its response for a parameter set corresponding to that used for the continuous-time model [13]. With over 99.9983% VAF the NARX output matched that of the continuous-time simulation with negligible error.

V. Identification of NARMAX coupled mass-spring-damper system model

In this section, we added at the outputs, $x_1\ (k)$ and $x_2\ (k)$, a Band-Limited White Noise $e_1\ (k)$ and $e_2\ (k)$, respectively. The NARMAX model is as follows:

$$\begin{aligned} x_{2,k} = &- \alpha_{1,k}\ x_{2,k-1} - \alpha_{2,k}\ x_{2,k-2} - \beta_{1,k}\ x_{1,k} \\ &+ \beta_{2,k}\ x_{1,k-1} - \beta_{3,k}\ x_{1,k-2} + \beta_{4,k}\ {x^3}_{1,k-2} \\ &+ \gamma_{1,k}u_{1,k-2} + \gamma_{2,k}u_{2,k-2} + \theta_{1,k}e_{2,k} \\ &+ \theta_{2,k}\ e_{2,k-1} + \theta_{3,k}\ e_{1,k} \end{aligned} \tag{16}$$

Fig. 5 presents the simulation input $x_2\left(k\right)$ and the predicted output of the NARMAX description model.

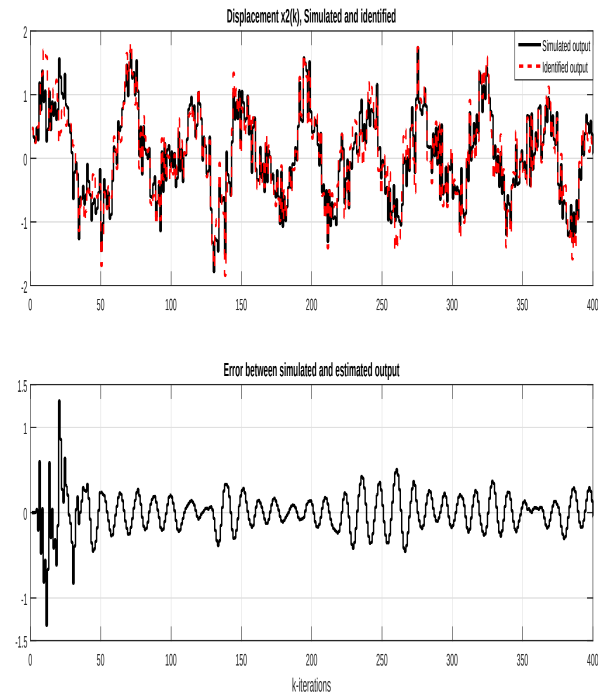

Fig. 5 – (top) Displacement $x_2\left(k\right)$, simulated and identified. (bottom) Error betwen simulated and identified output

Fig. 5 shows that the error between simulated and predicted output is important. Indeed, the predicted output matched the measured output with over 87.6601% VAF. The presence of perturbation on the NARMAX structure affects the quality of parametric estimation.

VI. Conclusion

The coupled mass-spring-damper can be represented by a higher order linear structure which complicate the elaboration of the optimal command. So, it is necessary to elaborate nonlinear structures to solve this problem. Indeed, based on the finite difference approximation, a NARX structure of coupled mass-spring-damper was obtained.

Simulation results showed that the estimated parameters of NARX structures are almost the same as the measured with negligible error. When a Band-Limited White Noise was added at the output, the quality of parametric estimation was be affeceted.

REFERENCES

[1] S. A. Billings and D. Coca, Identification of NARMAX and related models, *Control systems, Robotics and Automation*, vol. 6, pp. 110-139, 2004.

[2] A. Rahrooh and S. Shepard, Identification of nonlinear systems using NARMAX model, *Nonlinear Analysis*, vol. 71, pp. 1198-1202, 2009.

[3] I. J. Leontaritis and S. A. Billings, Inputoutput parametric models for nonlinear systems part I: Deterministic nonlinear systems, *International Journal of Control*, vol. 41, pp. 303328, 1985.

[4] I. J. Leontaritis and S. A. Billings, Inputoutput parametric models for nonlinear systems part II: Stochastic nonlinear systems, *International Journal of Control*, vol. 41, pp. 329344, 1985.

[5] S. L. Kukreja, H. L. Galiana and R.E. Kearney, NARMAX Representation and Identification of Ankle Dynamics. *IEEE Transactions On Biomedical Engineering*, vol. 50, pp. 70-81, 2003.

[6] O. Nelles, Nonlinear System Identification, *Springer*, 2001.

[7] S. Chen and A. Billings, Representations of non-linear systems: the NARMAX model *International Journal of Control*, vol. 50, pp. 70-81, 1989.

[8] T. H. Fay, Coupled spring equations, *International Journal of Mathematical Education in Science and Technology*, vol. 34, pp.65-79, 2003.

[9] I. R. Khan and R. Ohba, Closed-form expressions for the finite difference approximations of first and higher derivatives based on Taylor series, *Journal of Computational and Applied Mathematics*, vol. 107, pp. 179-193, 1999.

[10] F. Auger, Estimation par maximum de vraisemblace de la drive d'un signal bruit. Application la caractrisation de vrins pneumatiques, *19 Colloque sur le traitement du signal et des images*, pp. 758-761, 2003.

[11] A. Huseyin and P. Pramod, The Least Squares Algorithm, Parametric System Identification and Bounded Noise, *Automatica*, vol. 29, pp. 1535-1540, 1993.

[12] S. Sundari and A.Nachiappan, Online Identification Using RLS Algorithm and Kaczmarzs Projection Algorithm for a Bioreactor Process, *International Journal Of Engineering And Computer Science*, vol. 3, pp. 7974-7978, 2014.

[13] S. Ai, B. Mo, Z. Zheng and M. Yu, The Hardware-In-The-Loop simulation system of the diesel generator set based on the NARMAX model, *Proceedings of the 2nd International Conference on Computer Science and Electronics Engineering*, pp. 1080-1083, 2013.