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On the numerical differentiation problem of noisy signal

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Abstract— **This paper discusses the relevant theoretical problem of the numerical derivative estimation of noisy signals. In this paper, a comparative study of some different schemes of the differentiators is given: Kalman filter, the well-known Super Twisting algorithm, Super Twisting with dynamic gains and Euler backward difference method. The analysis of the study results can focus on the strengths and weaknesses of each algorithm with some chosen criteria.**

Keywords— numerical derivative, higher order sliding modes, Kalman filter; simulations.

I. Introduction

Numerical differentiation of measured signals is an old problem which is still considered as an attracted topic by many scientific communities and also by some industrial applications.

The major problem of noisy differentiation signal is the amplification of the noises on the signal estimation where it is so hard to discern between the noise and the basic signal.

This problem becomes more important in practice, where the measurement noises are not obvious to model.

Moreover, the estimation of any derivative can be destroyed even for a small high-frequency noises which is the usual assumption considered in most cases. Indeed, the practical differentiation consists to have a trade-off between exact differentiation and robustness with respect to noises.

In the literature, different approaches are proposed to deal with the problem of finding suitable linear or nonlinear algorithms able to reduce the noise effect while trying to leave the based signal with any phase shift.

In signal processing, many researchers have been developed. to cast the problem in terms of digital filter design frequency domain. Another standard approach is to invert the transfer function of a digital integrator well adapted to obtain a digital differentiator. One of the others traditional method is to use the finite difference method such as the Euler backward difference method. This one is simple to implement. However, with the sensors noises, this approach gives false results.

Although a low-pass filter can attenuate the noise on the estimation signals, some phase delay can be inevitable introduced.

Other approaches are directed towards an observer design problem where the knowledge of the system/noise model is necessary. In [1], the proposed differentiator is a high gain observer. The major drawback of this one is its sensitivity to measurement noises or perturbations which due to the infinity value of its gain.

In the case of random noises/perturbations, the Kalman filter [8] [9] can be used to estimate a derivative of some noisy signal which is optimal with the Gaussian white noises central is addressed by Kalman observers whose gains are computed by the resolution of an algebraic Ricatti equation [8].

In many cases the structure of the signal is unknown except from some differential inequalities, so, the observer approach can't be elaborated. An alternative approach is then addressed which consist of the design of differentiator scheme.

In such case, an algebraic differentiator [5] [6] [7] are based on truncated Taylor series of the signal to be differentiated are potentially interesting.

Other possible method is proposed which is relies on the higher order sliding mode [10] [11]. In [16], Levant proposes a 1st-order sliding mode differentiator which is well-known by "Super-Twisting". This last one is widely applied in the control context. Such algorithm has a simple form and is therefore easy to be implemented. However, its performance depends on the choice of the parameter values which are depended on the Lipschitz constant of the signal derivative. This constant is usually not known accurately beforehand, especially in the presence of noises. To avoid this problem, some research works were proposed a new scheme of the classical Super-Twisting, see for instance, [11-14].

In this paper, three kinds of differentiators are studied: Kalman algorithm [ref], Super-Twisting [16] and a new scheme of Super Twisting which have dynamic gains. By using an academic example, a detailed comparative study is given. This paper is organized as follows. Section 2 introduces the basic principle of the three differentiators. In section 3 presented an academic example. Section 4 is dedicated to discuss and to synthesize the obtained results.

II. Studied Differentiators

A. Kalman Filter

Started with the following state system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Mw(t) \\ y(t) = Cx(t) + Du(t) + v(t) \end{cases} \quad (1)$$

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Where x is the state vector, u and y are respectively the input and the output of the system.

A, B, C and D called respectively state matrix, input matrix, output matrix and feed-forward matrix of some system.

Where $x \in \Re^n; u \in \Re^m; y \in \Re^q$

$A \in \Re^{n\times n}; B \in \Re^{n\times m}; C \in \Re^{p\times n}; D \in \Re^{p\times m}$

w(t) and v(t) are stationary random signals and their covariance is zero. Note that W and V are respectively the Power Spectral Density of w(t) and v(t). In order to have an optimal behavior, the Kalman filter presents an assumption on W and V which must be Gaussian white noises central.

So the Kalman filter equations are as follow:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_f(Cx(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \quad (2)$$

With K_f is the gain of the filter and $\hat{x}$ is the estimate state.

The error expression can be taken as $\varepsilon(t) = x(t) - \hat{x}(t)$.

To minimize the error $\varepsilon(t)$, the gain K_f is so computed with using a defined follows criterion:

$$J(t) = E\left[\varepsilon^T(t)\varepsilon(t)\right] = traceE\left[\varepsilon(t)\varepsilon^T(t)\right] \quad (3)$$

With E is an expected value of $\left[\varepsilon^T(t)\varepsilon(t)\right]$ quantity.

Then with using the Riccati equation, the value of K_f is obtained as:

$$Kf = E\left[\varepsilon(t)\varepsilon^T(t)\right]C^T V^{-1} \quad (4)$$

B. Super TwistingDifferentiator

Consider $f(t)$ a measurable locally bounded function defined on $\left[0,\infty\right)$ as following:

$$f(t) = f_0(t) + \eta(t) \quad (5)$$

Where $f_0(t)$ is an unknown base signal with $(1+n)^{th}$ derivative having a known Lipschitz constant $C > 0$ and $\eta(t)$ is a bounded Lebesgue-measurable noise with unknown features, defined by: $\left|\xi(t)\right| < \varepsilon$ with ε is sufficiently small.

In [16], Levant defines an infinite number of differentiator schemes which makes possible to estimate the n^{th} derivative of the considered signal. So, from these schemes and for $n = 1$, a Super-Twisting differentiator (ST) can be defined by the following equations:

$$\begin{cases} \dot{z}_0 = v_0 \\ v_0 = z_1 - \lambda_0 \left|z_0 - f\right|^{\frac{1}{2}} \operatorname{sgn}(z_0 - f) \\ \dot{z}_1 = v_1 \\ v_1 = -\lambda_1 \operatorname{sgn}(z_0 - v_1) \end{cases} \quad (6)$$

Where z_0 and z_1 are the states of the algorithm and v_0 is its output.

Under sufficient conditions given in [16], λ_0, λ_1 are the positive gains ensuring a finite-time convergence of the algorithm by satisfying these inequalities:

$$\begin{aligned} &\lambda_1 > C \\ &\lambda_0^2 \geq 4C\frac{\lambda_1 + C}{\lambda_1 - C} \end{aligned} \quad (7)$$

According to this expression, the gain choice assumes the knowledge in advance of the Lipschitz constant of the (n+1) order derivative of the useful signal. But, if the signal is noisy, the Lipschitz constant is a priori unknown and the choice becomes even more difficult.

Therefore, the main drawback of such algorithm is the setting of its gains to keep good performances even when the frequency of the input signals of differentiator changes or if the input spectrum has rich frequencies. It is not always easy to tune the parameters values λ_i for a given bandwidth of the input signal.

From (6), it can be seen that the terms z_i introduce the integral components and act as estimators of the input signal derivative. Theoretically, the ideal sliding mode must ensure the first terms of equations (6) to zero in finite time. However, the ideal sliding mode never be realized in the practice owing to the different origin of inaccuracy, such as a measurement errors. Moreover, the presence of “sign(.)” function in these terms leads to high frequency oscillations. Indeed, this chattering effect can deteriorate the precision of the estimated signal. Therefore, it is not easy to adjust the gains to reach a good compromise between accuracy and robustness to noise ratio.

Then, to avoid this problem, a new differentiator is proposed called Dynamic Gains Super Twisting differentiator.

C. Dynamic Gains Super Twisting differentiator

In [14], the Dynamic Gains Super Twisting (DGST) differentiator is proposed to facilitate the adjustment of differentiating gains while providing a good compromise between accuracy and noises robustness.

This new scheme is based on adding a linear function to the second term of the algorithm.

This algorithm keeps the same notations as the classic ST.

The proposed differentiator is given by the following system, where f (t) is an input signal of the algorithm:

$$\begin{cases} \dot{z}_0 = v_0 \\ v_0 = -\hat{\lambda}_0 \left|z_0 - f\right|^{\frac{1}{2}} sign(z_0 - f) - k_0(z_0 - f) + z_1 \\ \dot{z}_1 = v_1 \\ v_1 = -\hat{\lambda}_1 sign(z_0 - f) \end{cases} \quad (8)$$

Where k_0 is the convergence positive gain of the differentiator.

The dynamic gains $\hat{\lambda}_0, \hat{\lambda}_1$ are defined by:

$$\begin{cases} \dot{\hat{\lambda}}_0 = \left[|\lambda_0|^{\frac{1}{2}} sign(\lambda_0) \right] \lambda_0; \ \ \hat{\lambda}_0(0) \geq 0 \ \ and \ \ \dot{\hat{\lambda}}_0 \succ 0; \forall t \succ 0 \\ \dot{\hat{\lambda}}_1 = |\lambda_0|; \ \ \hat{\lambda}_1(0) \geq 0 \ \ and \ \ \dot{\hat{\lambda}}_1 \succ 0; \forall t \succ 0 \end{cases} \quad (9)$$

For the proof convergence of the algorithm, see [15].

III. Academic Example: Mechanical system "mass, spring, damper"

To study the effectiveness of the presented differentiators, a mechanical system is used; see (Fig.1) to compute speed of the mass m.

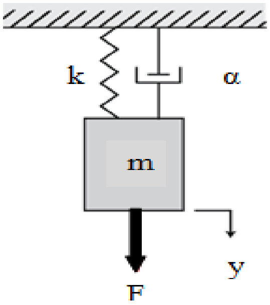

Fig.1. Mass- Spring- Damper System

The mathematical model of the system is as follows:

$$m\ddot{y}(t) + \alpha\dot{y}(t) + ky(t) = F(t) \quad (10)$$

Where: k : spring stiffness (N / m); α : spring damping (Ns / m); m : mass of the object (Kg); $F(t)$: external force(N).

Then the state representation is written as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

With:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\alpha}{m} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (11)$$

IV. Simulation Results and Analysis

Let consider the noisy input signal differentiator shown in (Fig.2). For the simulation tests, the SIMULINK is used with sampling period T_e equal to 10^{-3} s. The parameters of the system are chosen as: $m = 1kg, k = 1N.m^{-1}, \alpha = 0.5N.s.m^{-1}$.

The parameter of each algorithm is taken as: the gain of the Kalman filter $K_f = \begin{bmatrix} 0.4255 \\ 0.0905 \end{bmatrix}$. The Super-Twisting differentiator parameters are: $\lambda_0 = 4, \lambda_1 = 7$. The value of the Dynamic Gains Super Twisting gain is $k_0 = 6$.

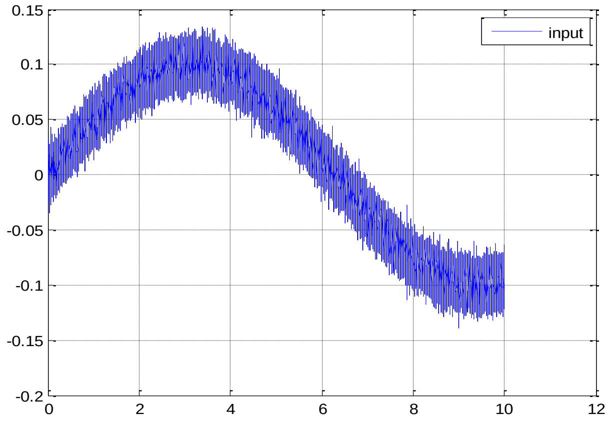

Fig. 2. Input signal : position y(t) of the mass

The comparison criteria adopted is the absolute value of magnitude error and the delay (pick to pick) between the signals.

For the estimation of the first derivative in presence of Gaussian white central noise, the simulations show that:

The (fig.3.a) shows the Super-Twisting differentiator presents a maximum error 10 times more than that given by the Dynamic Gains Super Twisting differentiator this big diffenrence is because the presence of a linear part $k_0(z_0 - f)$ in the anallytic expression of Dynamic Gains Super Twisting differentiator; the adding of this continuous term enssures smoothing of noise at the output thanks to the low value of convergence gains.

The maximum error of the Dynamic Gains Super Twisting differentiator is more important than that given by kalman filter; in contrast if we add a filtering stage on the algorithm of Gains Super Twisting differentiator the accuracy will be improved. For the phase shift, the Kalman filter has the lowest phase schift.

Finally, the Kalman filter presents the best result because it puts in their optimal conditions that is to say with Gaussian white central noise. Or in the opposite case (with other type of noise) the Dynamic Gains Super Twisting differentiators may be presents more efficient results.

Table.I summarizes the results obtained during the simulations.

TABLE I
ESTIMATION ERRORS AND PHASE SHIFT FOR THE ESTIMATE OF THE FIRST DERIVATIVE

Algorithm	**KALMAN**	**ST**	**DGST**
$\lvert e_{max} \rvert$	0.00865	0.01976	0.009803
Phase shift (°)	0.079	0.140	0.106

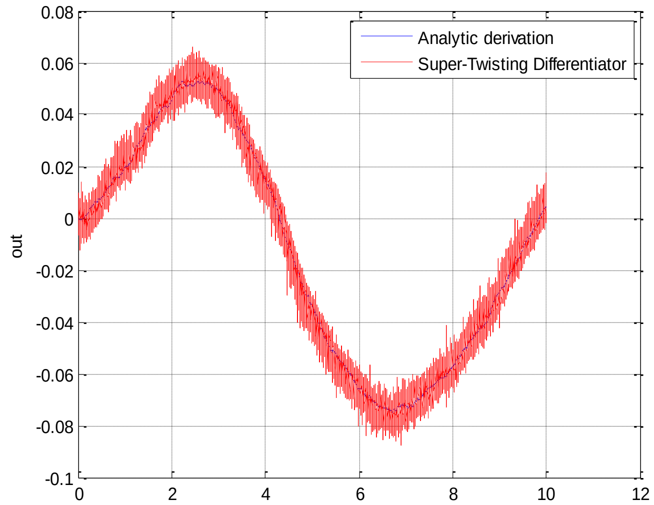

Fig.3.a. Super Twisting differentiator

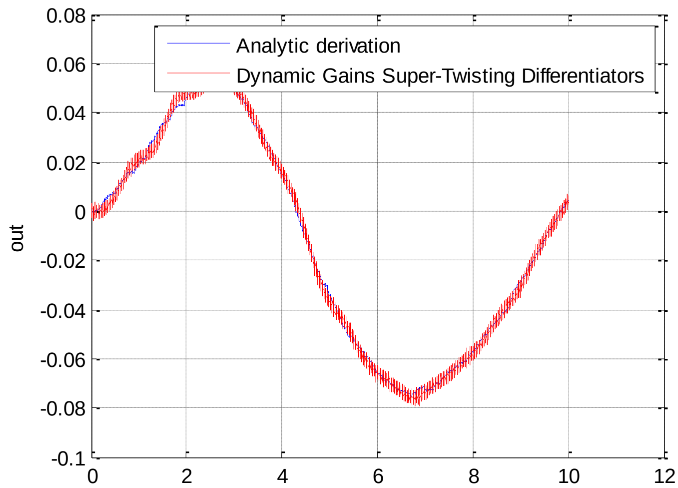

Fig. 3.b. Dynamic Gains Super Twisting differentiator

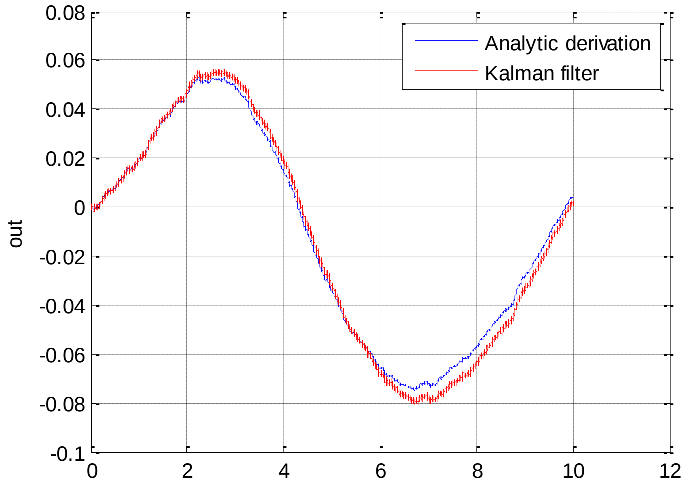

Fig.3.c. Kalman differentiator

Fig. 3. Differentiator outputs

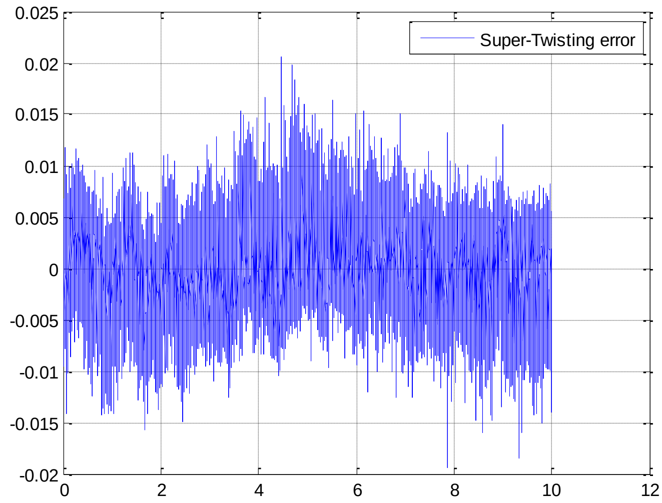

Fig. 4.a. Super Twisting error

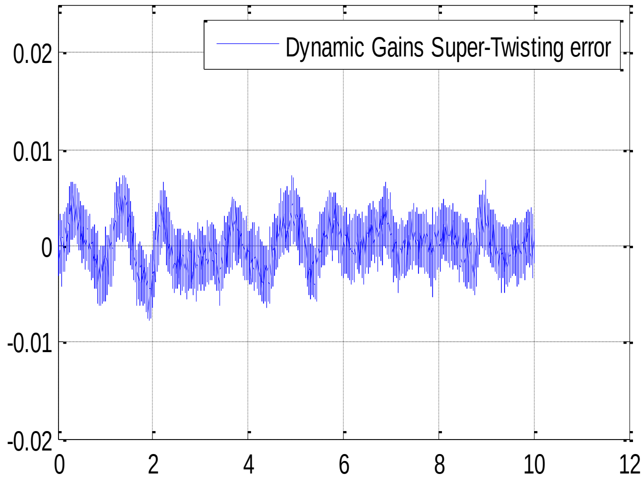

Fig.4.b. Dynamic Gains Super Twisting error

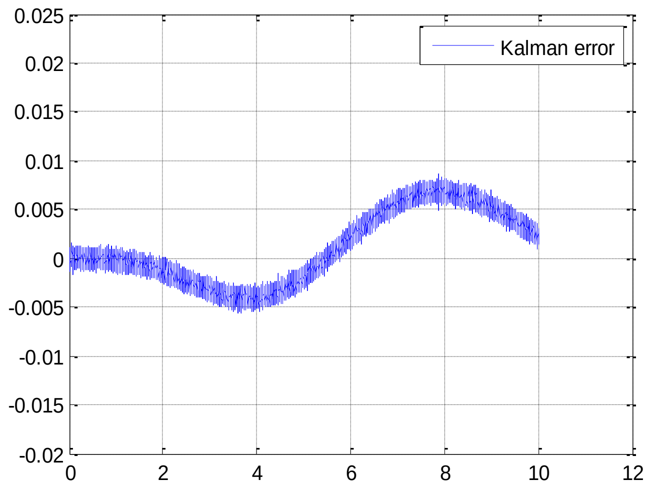

Fig.4.c. Kalman error

Fig.4. Error curve

In table II, the different simulation results provided in this paper for the different algorithms differentiation are given. A comparative summary was compiled to highlight the advantages and disadvantages of each method

TABLE II
COMPARISON OF DIFFERENT SCHEMAS OF DIFFERENTIATION

Algorithm	precision	phase shift	robustness	ease of implementation	calculation time
ST	+	-	-	-	-
DGST	++	+	+	+	++
KALMAN	+++	++	+	++	--

V. CONCLUSION

In this paper we proved that the adjustment of the Kalman filter is not easy. Indeed, the optimality of its accuracy is ensured that despite very strong assumption. In practice, various perturbations signals are not necessary to be white. The Dynamic Gains Super Twisting differentiator may give better results than the kalman filter in the case adds a filter to it.

It is possible to use cascade algorithms in order to compute acceleration of the mass. In th future works, it is an important to study these differentiators for different kinds of noises and also input signals.

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