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State Equality Linear Constraint Enforcement on the Linear Quadratic Regulator

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***Abstract*—The Linear Quadratic Regulator (LQR) optimal control problem with linear state equality constraint enforcement is investigated for continuous time-invariant systems. It is shown that a control vector that complies with a constraint equation must be composed of two components that act on two orthogonally complement subspaces of the control space. The state equality constraint enforcement on the LQR optimal control policy takes place exclusively by only one of the two components. Moreover, this component is unique and is independent from the quadratic cost functional optimization that is performed exclusively by the other component of the control vector. The problem is reformulated accordingly by directly augmenting the constraint with the unconstrained system dynamics using the constraint-enforcing component of the control vector, which reduces the differential-algebraic form of constrained system dynamics to an equivalent differential form. The optimal feedback control law is then obtained by solving a classical matrix algebraic Riccati equation. The present approach forms a departure from the Lagrange multiplier technique of augmenting algebraic constraints with integrands of cost functionals. Application to partial Eigenstructure assignment on the LQR optimal control problem is conducted.**

I. Introduction

The continuous-time Linear Quadratic Regulator (LQR) was introduced by [1] and [2]. Its development was a breakthrough for multiple-input, multiple-output (MIMO) control systems because it provided an optimal and systematic manner of control variables coordination to the MIMO control system. This "automatic" coordination was missing from the previous "successive loop-closure" approach to MIMO control system design, because that approach can easily give results that are far from optimum, e.g., poorly coordinated controls that fight each other, thus wasting control authority [3].

Beside alleviating the control variables coordination problem, LQ regulators and tracking compensators enjoy favorable inherent robustness characteristics. In particular, the corresponding closed loop systems guarantee gain and phase margins of at least $(-6,\infty)$ db and (-60,60) degrees, respectively, see, e.g., [4] pp. 383.

Nevertheless, an important extension that has been missing from the literature of continuous-time LQR is state equality constraint enforcement on the closed loop control system. Such an extension is important because many control system performance requirements can be casted in the form of equations that involve some or all of the state variables. The endeavour is taken in this paper to pursue the LQR control law for a linear time invariant (LTI) system while a linear state-equality constraint is to be enforced on its closed loop dynamics.

The present approach for solving the state equality constrained LQR optimal control problem is based on the observation that the constraint enforcement is performed exclusively by one part of the control vector, with no interference from the other part. The constraint-enforcing part is the component that acts the range of the transposed coefficient of the control vector in the constraint equation after evaluating it along the state trajectories of the LTI system. Moreover, the constraint-enforcing component is unique and must be independent from the cost functional optimization, which is performed by the remaining part of the control vector, i.e., the component that is in the nullspace of the controls coefficient vector.

Accordingly, generalized dynamic inversion (GDI) of the algebraic constraint equation is performed using the Greville formula [5], which is based on the Moore-Penrose generalized inverse [6], [7]. The GDI closed loop control system exhibits an advantageous geometric structure [8], [9], which allows to separate the constraint enforcement control task from the cost functional optimization task. Moreover, it is shown in this paper that applying GDI with the LQR optimal control on a linear system provides an eigenstructure to the closed loop system. This structure is missing from the LQR closed loop control system when the LQR is applied in its standard formulation. The performance of the proposed constrained LQR control design is illustrated on the control of the lateral dynamics of a general transport aircraft.

II. Linear State Constraint Enforcement on LQR Design

Consider the LTI system in the following state space form

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and control vectors, and the constant matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the system and control, respectively. It is assumed that the pair (A,B) is stabilizable and that the state vector x is fully measurable and available for feedback. It is required to design a feedback

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control law $u(x,t)$ that minimizes the infinite horizon quadratic cost functional

$$J = \int_0^\infty (x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u) dt \tag{2}$$

while forcing the following linear state-constraint on the closed loop system

$$Dx = d(t) \tag{3}$$

where $D \in \mathbb{R}^{1\times n}$, and $d : [0,\infty) \to \mathbb{R}$. The state weighting matrix $Q \in \mathbb{R}^{n\times n}$ in the expression of J is constant nonnegative, and the control weighting matrix $R \in \mathbb{R}^{m\times m}$ is constant positive definite. The pair $(A, \sqrt{Q})$ is assumed detectable, and the function $d(t)$ is assumed to be at least r-times differentiable, where r is the relative degree of Dx with respect to u. Hence, an error variable e is defined as

$$e = Dx - d(t) \tag{4}$$

and the following desired error dynamics is prescribed

$$k_0 e^{(r)} + k_1 e^{(r-1)} + \cdots + k_{r-1}\dot{e} + k_r e = 0 \tag{5}$$

where the constants $k_0, \ldots, k_r$ are chosen such that the equilibrium point $e = 0$ of (5) is asymptotically stable. Evaluating the first r time derivatives of e along the solution trajectories of (1) yields

$$e^{(i)} = \begin{cases} DA^i x - d^{(i)}(t) & i = 1 \ldots r-1 \\ DA^r x + DA^{r-1}Bu - d^{(r)}(t) & i = r. \end{cases} \tag{6}$$

Substituting (6) in (5) gives rise to the following algebraic form of the differential error dynamics

$$\mathscr{A} u = \mathscr{B}(x,t) \tag{7}$$

where the *controls coefficient* $\mathscr{A} \in \mathbb{R}^{1\times m}$ is

$$\mathscr{A} = k_0 D A^{r-1} B \tag{8}$$

and the *controls load* $\mathscr{B} : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is

$$\mathscr{B}(x,t) = K_1 x + v(t) \tag{9}$$

where the feedback gain $K_1 \in \mathbb{R}^{1\times n}$ is

$$K_1 = -D \sum_{j=0}^{r} k_j A^{r-j} \tag{10}$$

and the feedforward command $v : [0,\infty) \to \mathbb{R}$ is

$$v(t) = \sum_{j=0}^{r} k_j d^{(r-j)}(t). \tag{11}$$

The GDI control law that enforces the error dynamics (5) is given by [5]

$$u = \underbrace{\mathscr{A}^+ \mathscr{B}(x,t)}_{\in \mathbf{R}(\mathscr{A}^{\mathrm{T}})} + \underbrace{\mathscr{P} u_a}_{\in \mathbf{N}(\mathscr{A})} \tag{12}$$

where $\mathscr{A}^+ \in \mathbb{R}^m$ in the first (particular) part of the GDI control law is the MPGI of $\mathscr{A}$, uniquely satisfying the four Penrose equations [7], [11] pp. 40

$$\mathscr{A}\mathscr{A}^+\mathscr{A} = \mathscr{A} \tag{13}$$
$$\mathscr{A}^+\mathscr{A}\mathscr{A}^+ = \mathscr{A}^+ \tag{14}$$
$$(\mathscr{A}\mathscr{A}^+)^{\mathrm{T}} = \mathscr{A}\mathscr{A}^+ \tag{15}$$
$$(\mathscr{A}^+\mathscr{A})^{\mathrm{T}} = \mathscr{A}^+\mathscr{A} \tag{16}$$

and is given for the single row vector $\mathscr{A}$ as

$$\mathscr{A}^+ = \frac{\mathscr{A}^{\mathrm{T}}}{\mathscr{A}\mathscr{A}^{\mathrm{T}}}. \tag{17}$$

The vector $\mathscr{A}^+$ maps $\mathscr{B}(x,t)$ to $\mathbf{R}(\mathscr{A}^{\mathrm{T}})$ (range space of $\mathscr{A}^{\mathrm{T}}$), and the *null-projection matrix* $\mathscr{P} : \mathbb{R}^{1\times m} \to \mathbb{R}^{m\times m}$ in the second (auxiliary) part of the GDI control law is given by

$$\mathscr{P} = I_{m\times m} - \mathscr{A}^+\mathscr{A} \tag{18}$$

and it maps the *null-control vector* $u_a \in \mathbb{R}^m$ to $\mathbf{N}(\mathscr{A})$ (nullspace of $\mathscr{A}$) [11] pp. 374.

It follows from (17) and (18) that $\mathscr{A}\mathscr{A}^+ = 1$ and $\mathscr{A}\mathscr{P} = 0$. Therefore, multiplying both sides of (12) by $\mathscr{A}$ recovers the algebraic constraint given by (7) irrespective of u_a. Hence, u_a is arbitrary as for the purpose of enforcing the differential error constraint given by (5). In particular, this constraint would be enforced by simply setting $u_a = 0_m$, which gives the "minimum norm solution" of u, but might render the GDI closed loop control system unstable. This also implies that the constraint enforcement is achieved solely by means of the (unique) particular part of the GDI control law (12), and that it is independent from the cost functional optimization, which must take place in $\mathbf{N}(\mathscr{A})$ by means of u_a.

Substituting (12) in (1) yields the following GDI control system

$$\dot{x} = Ax + B(\mathscr{A}^+\mathscr{B}(x,t) + \mathscr{P} u_a). \tag{19}$$

Notice that because u_a is free, only the particular control loop is closed in (19), and u_a parameterizes all LTI control systems in the form (1) that are constrained by a differential error constraint in the form given by (5).

The "half-closed" GDI control system (19) is now rewritten as

$$\dot{x} = Ax + B_p \mathscr{B}(x,t) + B_a u_a \tag{20}$$

where $B_p \in \mathbb{R}^{n\times 1}$ and $B_a \in \mathbb{R}^{n\times m}$ are the particular and the auxiliary control matrices given by $B\mathscr{A}^+$ and $B\mathscr{P}$, respectively. The range spaces $\mathbf{R}(B_p)$ and $\mathbf{R}(B_a)$ are related to $\mathbf{R}(\mathscr{A}^{\mathrm{T}})$ and $\mathbf{N}(\mathscr{A})$ by the equations

$$\mathbf{R}(B_p) = \mathbf{R}(B\mathscr{A}^+) = B\mathbf{R}(\mathscr{A}^+) = B\mathbf{R}(\mathscr{A}^{\mathrm{T}}) \tag{21}$$

and

$$\mathbf{R}(B_a) = \mathbf{R}(B\mathscr{P}) = B\mathbf{R}(\mathscr{P}) = B\mathbf{N}(\mathscr{A}). \tag{22}$$

Hence, while B maps a control vector $u \in \mathbb{R}^m$ to $\mathbf{R}(B)$, B_p maps only the projection of u onto $\mathbf{R}(\mathscr{A}^{\mathrm{T}})$ to $\mathbf{R}(B)$, and B_a maps only the projection of u onto $\mathbf{N}(\mathscr{A})$ to $\mathbf{R}(B)$. Nevertheless, because $\mathbf{R}(\mathscr{A}^{\mathrm{T}})$ and $\mathbf{N}(\mathscr{A})$ are orthogonally complement subspaces of $\mathbb{R}^m$, it follows that the control space that is spanned by the columns of B is partitioned into two control subspaces that are spanned by the columns of B_p and B_a. Moreover, evaluating $B_a^{\mathrm{T}} B_p$ yields

$$B_a^{\mathrm{T}} B_p = \mathscr{P} B^T B \mathscr{A}^+ = 0_m \Leftrightarrow B^{\mathrm{T}} B = I_{m\times m}. \tag{23}$$

Hence, the two control subspace partitions are orthogonally complements if and only if the columns of B are orthonormals.

The controllability matrix of the particular GDI control subsystem is given by

$$\begin{aligned} \mathrm{C}_p &= \begin{bmatrix} B_p & AB_p \cdots A^{n-1}B_p \end{bmatrix} && (24)\\ &= \begin{bmatrix} B & AB \cdots A^{n-1}B \end{bmatrix} bdiag(\mathscr{A}^+) && (25)\\ &= \mathrm{C}\, bdiag(\mathscr{A}^+) && (26) \end{aligned}$$

where $\mathrm{C} \in \mathbb{R}^{n\times mn}$ is the controllability matrix of system given by (1), and $bdiag(\mathscr{A}^+) \in \mathbb{R}^{mn\times n}$ is the block-diagonal matrix that contains $\mathscr{A}^+$ as each of its block diagonal elements. The controllability subspace of the particular GDI control subsystem is

$$\begin{aligned} \mathscr{C}_p &= \mathbf{R}(\mathrm{C}_p) = \mathbf{R}(\mathrm{C}\, bdiag(\mathscr{A}^+)) && (27)\\ &= \mathrm{C}\, \mathbf{R}(bdiag(\mathscr{A}^+)). && (28) \end{aligned}$$

Hence, $\mathscr{C}_p$ that is spanned by the columns of C_p is a subspace of the controllability subspace $\mathscr{C}$ of the system (1), and is obtained by restricting each of the n block columns of C to map only the vectors in $\mathbf{R}(\mathscr{A}^T)$ to $\mathscr{C}$.

Similarly, the controllability matrix of the auxiliary GDI control subsystem is given by

$$\begin{aligned} \mathrm{C}_a &= \begin{bmatrix} B_a & AB_a \cdots A^{n-1}B_a \end{bmatrix} && (29)\\ &= \begin{bmatrix} B & AB \cdots A^{n-1}B \end{bmatrix} bdiag(\mathscr{P}) && (30)\\ &= \mathrm{C}\, bdiag(\mathscr{P}) && (31) \end{aligned}$$

where $bdiag(\mathscr{P}) \in \mathbb{R}^{mn\times mn}$ is the block-diagonal matrix that contains $\mathscr{P}$ as each of its block diagonal elements. The controllability subspace of the auxiliary GDI control subsystem is

$$\begin{aligned} \mathscr{C}_a &= \mathbf{R}(\mathrm{C}_a) = \mathbf{R}(\mathrm{C}\, bdiag(\mathscr{P})) && (32)\\ &= \mathrm{C}\, \mathbf{R}(bdiag(\mathscr{P})). && (33) \end{aligned}$$

Hence, $\mathscr{C}_a$ that is spanned by the columns of C_a is a subspace of $\mathscr{C}$, and is obtained by restricting each of the n block columns of C to map only the vectors in $\mathbf{R}(\mathscr{P})$ to $\mathscr{C}$.

III. Constrained LQR Control Design

Substituting the expression of $\mathscr{B}(x,t)$ given by (9) in (19) yields

$$\begin{aligned} \dot{x} &= Ax + B_p(K_1 x + v(t)) + B_a u_a && (34)\\ \dot{x} &= A_{c_p} x + \underbrace{B_p v(t)}_{\in B\mathbf{R}(\mathscr{A}^{\mathrm{T}})} + \underbrace{B_a u_a}_{\in B\mathbf{R}(\mathscr{P})} && (35) \end{aligned}$$

where $A_{c_p} = A + B_p K_1$. Because $v(t)$ is mapped to $\mathbf{R}(\mathscr{A}^{\mathrm{T}})$, which is normal to the action space of u_a, the dynamics of the second term is completely uncontrollable by u_a. Therefore, the second term is independent from the cost functional optimization, and the goal reduces to a classical LQR design of the null-control vector u_a such that the following performance index is minimized

$$J = \int_0^\infty (x^{\mathrm{T}} Q x + u_a^{\mathrm{T}} R u_a) dt \tag{36}$$

where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are such that $Q \geq 0$ and $R > 0$, and subject to the dynamic constraint

$$\dot{x} = A_{c_p} x + B_a u_a. \tag{37}$$

The solution is $u_a = K_a x$, where $K_a = -R^{-1} B^T P$, and P is the solution of the algebraic Riccati equation (ARE) [10]

$$A_{c_p}^{\mathrm{T}} P + P A_{c_p} - P B R^{-1} B^T P + Q = 0. \tag{38}$$

IV. Application to Eigenstructure Assignment

The LQR compensator is well known for its favorable robustness characteristics. However, the standard LQR lacks the eigenstructure assignment property, which makes shaping the closed loop response in a systematic manner an unreachable goal, and leaves it to experience and knowledge of the physical system to determine the quadratic weighting matrices, which often turns out to become a trial and error procedure.

A few attempts were recorded in the LQR control literature to equip the methodology with the eigenstructure capability, e.g., [12], [13], [14]. All these modifications aim to determine the weighting matrices that yield closed loop systems with desired sets of eigenvalues/eigenvectors. However, iterating between different weighting matrices deviates from the essence of optimal control, which aims at optimizing a predetermined cost functional. An application of the present design in partial eigenvalue assignment problem is presented.

A. Example 1: Partial Eigenvalue assignment of Lateral Aircraft Dynamics

The lateral dynamics of a transport aircraft is approximated by the following state space model

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{\phi} \\ \dot{p} \end{bmatrix} = 10^{-2} \begin{bmatrix} -10.0 & -100.0 & 11.5 & 0 \\ 40.9 & -24.5 & 0 & -4.0 \\ 0 & 0 & 0 & 100.0 \\ -160.4 & 28.5 & 0 & -109.3 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ \phi \\ p \end{bmatrix} + 10^{-2} \begin{bmatrix} 0 & 1.8 \\ -0.2 & -24.4 \\ 0 & 0 \\ 32.2 & 8.7 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \tag{39}$$

where β is the side slip angle, r is the yaw rate, ϕ is the roll angle, p is the roll rate, δ_a and δ_r are respectively the aileron and the rudder deflections. Hence, the state and control vectors are defined as $x = \begin{bmatrix} \beta & r & \phi & p \end{bmatrix}^{\mathrm{T}}$ and $u = \begin{bmatrix} \delta_a & \delta_r \end{bmatrix}^{\mathrm{T}}$. The eigenvalues of the system matrix are

$$\begin{aligned} \lambda_r &= -1.2308 && (40)\\ \lambda_{d,\bar{d}} &= -0.0806 \pm 0.7433i && (41)\\ \lambda_s &= -0.0464 && (42) \end{aligned}$$

where λ_r is the highly damped first order roll subsidence mode eigenvalue, λ_d and $\lambda_{\bar{d}}$ are the Dutch roll's oscillatory mode eigenvalues, and mainly represent lightly damped β and r oscillations. The eigenvalue λ_s is the spiral mode eigenvalue, and is dominated by r. The corresponding set of eigenvectors is

$$v_r = 10^{-2}\begin{bmatrix} 6.7 \\ 0.3 \\ -62.9 \\ 77.4 \end{bmatrix}, \; v_{d,\bar{d}} = 10^{-2}\begin{bmatrix} 24.7 \pm 33.4i \\ -17.2 \pm 17.7i \\ 70.1 \\ -5.7 \pm 52.1i \end{bmatrix},$$

$$v_s = 10^{-2}\begin{bmatrix} 5.0 \\ 11.2 \\ 99.1 \\ -4.6 \end{bmatrix}.$$

Since λ_s is almost neutrally stable, it is desired to enhance the the spiral mode characteristics by removing λ_s further to the left on the real line. Hence, the following desired yaw dynamics is prescribed

$$\dot{r} - \lambda_{s_d} r = 0 \tag{43}$$

where λ_{s_d} is a desired value of λ_s. Evaluating the desired yaw dynamics (43) along the trajectory solutions of the state space model (39) yields (9), where $p = 1$, $m = 2$, and $r(t) = 0$. The resulting time invariant algebraic equation is

$$\mathscr{A} u = K_1 x \tag{44}$$

where

$$\mathscr{A} = \begin{bmatrix} -0.0017 & -0.2440 \end{bmatrix} \tag{45}$$

and

$$K_1 = \begin{bmatrix} -0.4089 & 0.2454 + \lambda_{s_d} & 0 & 0.0395 \end{bmatrix}. \tag{46}$$

Therefore, the GDI control law given by (12) becomes

$$u = \mathscr{A}^+ K_1 x + \mathscr{P} u_a \tag{47}$$

where

$$\mathscr{A}^+ = \frac{\mathscr{A}^{\mathrm{T}}}{\mathscr{A}\mathscr{A}^{\mathrm{T}}} = \begin{bmatrix} -0.0286 \\ -4.0982 \end{bmatrix} \tag{48}$$

and

$$\mathscr{P} = \begin{bmatrix} 1 & -0.007 \\ -0.007 & 0 \end{bmatrix}. \tag{49}$$

The aircraft's lateral dynamics after closing the particular part of the GDI control law is obtained by substituting (47) in (39), resulting in

$$\dot{x} = A_{c_p} x + B_a u_a \tag{50}$$

where

$$A_{c_p} = A + B_p K_1 =$$
$$10^{-2}\begin{bmatrix} -6.9 & -101.8 - 7.5\lambda_{s_d} & 11.5 & -0.29 \\ 0 & 100\lambda_{s_d} & 0 & 0 \\ 0 & 0 & 0 & 100 \\ -145.5 & 19.5 - 36.5\lambda_{s_d} & 0 & -110.7 \end{bmatrix}. \tag{51}$$

The eigenvalues of A_{c_p} are

$$\lambda_r = -1.2290 \tag{52}$$
$$\lambda_{d,\bar{d}} = 0.0260 \pm 0.3685i \tag{53}$$
$$\lambda_s = \lambda_{s_d}. \tag{54}$$

$$v_r = \begin{bmatrix} 0.0834 \\ 0 \\ -0.8137 \\ 1 \end{bmatrix}, \; v_{d,\bar{d}} = \begin{bmatrix} -0.7793 \mp 0.2533i \\ 0 \\ 0.1903 \mp 2.700i \\ 1 \end{bmatrix}, \tag{55}$$

$$v_s = \begin{bmatrix} \frac{3.73\lambda_{s_d}^3 + 55\lambda_{s_d}^2 + 58.5\lambda_{s_d} - 1.1265}{(18.25\lambda_{s_d}^2 - 13.93\lambda_{s_d} - 74.74)\lambda_{s_d}} \\ \frac{-50\lambda_{s_d}^3 - 58.85\lambda_{s_d}^2 - 3.63\lambda_{s_d} - 8.39}{(18.25\lambda_{s_d}^2 - 13.93\lambda_{s_d} - 74.74)\lambda_{s_d}} \\ \frac{1}{\lambda_{s_d}} \\ 1 \end{bmatrix}. \tag{56}$$

The second row of A_{c_p} reflects the enforcement of the desired yaw dynamics (43) on the feedback control system that is achieved by closing the particular loop of the GDI control law, such that the spiral mode is assigned its desired value λ_{s_d}. Moreover, the effects of the roll subsidence and Dutch roll modes on the time response of r are eliminated as depicted from the second element of v_r and the second elements of $v_{d,\bar{d}}$ which are both zeros in accordance to satisfying the constraint (43), implying spiral mode decoupling.

Closing the particular loop of the GDI control law has also minor effects on the roll subsidence and Dutch roll modes, as seen by comparing (40-42) with (52-54). It is noticed for instance that closing the loop of the particular part of the GDI control law has very minor effects on stability of the roll and Dutch roll modes, and it reduces the frequency of the Dutch roll mode to half of its open loop value. To enhance stability of the Dutch roll mode, the null-control can be designed using LQR, for instance to optimize the functional

$$J = \int_0^\infty (x^{\mathrm{T}} x + u_a^{\mathrm{T}} u_a) dt. \tag{57}$$

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