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Causal links between Bond and Stock Market: Crushes, contagion, and bubbles

Rabeb Mahjoub *[1], Ali Trabelsi Karoui [2], Aida Kammoun [3]

[1] Doctor of Finance, Faculty of Economics, and management science of Sfax; rabeb.mahjoub@gmail.com

[2] Doctor of Finance, Faculty of Economics, and management science of Sfax; ali.trabelsikaroui@fsegma.u-monastir.tn

[3] Professor of Finance, Higher Institute of Business Administration of Sfax; aida.kammoun@isaas.rnu.tn

* Corresponding author. E-mail: rabeb.mahjoub@gmail.com

Abstract:

The primary objective of this study is to investigate the causal links and risk contagion mechanisms between European stock and bond markets, particularly during crises. It seeks to determine the origins of market crashes and the evolving dynamics of contagion over time. By examining these relationships, the research aims to provide insights that help investors and policymakers mitigate systemic risks and develop effective strategies for managing contagion across asset classes in interconnected markets. The research employs four contagion tests to analyze risk transmission between European stock and bond markets during crisis and non-crisis periods. By correlating recurring crash times with significant events, it identifies causal links and sources of crashes. The methodology focuses on comparing the systemic risk dynamics during the financial and COVID-19 crises to assess how markets respond to shocks. This robust, data-driven approach sheds light on the bidirectional nature of contagion and the influence of speculative bubbles. The study confirms a substantial contagion effect between European stock and bond markets, with dynamics varying over time, especially during crises. It reveals that systemic risk transmission during the COVID-19 crisis was less significant than during the financial crisis, suggesting market adaptation to past crises. Contagion is bidirectional, emphasizing the critical role of stock market bubbles in bond market crashes and the origin of crises. The findings underline the stock market's catalytic role in contagion and its importance for understanding risk propagation in interconnected financial systems. This study offers a unique contribution by examining the evolving

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dynamics of risk contagion between European stock and bond markets across different crises, including the financial and COVID-19 crises. Employing four contagion tests, it highlights the bidirectional nature of contagion and underscores the role of speculative bubbles in stock markets as a catalyst for bond market crashes. The research provides actionable insights for investors to better understand risk transmission mechanisms and develop strategies to mitigate contagion effects, combining both asset classes.

Keywords:
Crashes, Contagion, Stock Market, Risk Transmission, Government Bonds, Bubbles

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1. Introduction and literature review

As the volatility of sovereign debt spreads increases, so does the level of sovereign risk. This phenomenon possesses the capacity to influence government regulation, political stability, exchange rates, interest rates, and other related factors. Sovereign risk can also be linked to the stock market catastrophe. In the fields of economics and finance, a contagion refers to a situation where a disturbance in one economy or region propagates and affects another. Financial crises can give rise to contagions, which are negative externalities that propagate from one collapsing market to another. According to Brana and Lahet (2005), contagion refers to a scenario in which the impact of an external shock exceeds the expectations based on fundamental factors.

Kaminsky and Reinhart (1999) provide evidence for the occurrence of a contagion phenomenon in a particular crisis, while Forbes and Rigobon (2002) and Billio and Pelizzon (2003) challenge the existence of a contagion phenomenon by explaining the transmission of shocks because of regular interdependencies between countries. In addition, Masson (1998) delineates three pivotal factors contributing to the dissemination of infection. There is no mutual influence between countries because of lunar or external factors. The second form of interdependence is known as normal interdependencies, which pertain to the customary transfer of effects between markets or interdependencies. The last category is characterized by pure contagion, wherein the failure or disaster of a single country has a widespread impact on the entire market. Our research examines all three types of contagion, analyzing a range of accidents and their causes, both domestic and international.

Beirne and Marcel (2013) elucidate the concept of sovereign risk by examining the significance of spillovers, contagion of sovereign yield spreads, and CDS spreads in the context of the European sovereign debt crisis. Their discoveries substantiate the phenomenon of herding contagion. Allen and Gale (2000) define "contagion" as the phenomenon of excessive spillover, illustrating how a financial crisis in one location can propagate to other locations. Jean (2012) conducts quantitative research on financial markets to highlight the interconnections between key asset classes, including stocks, bonds, and credit default swaps (CDS). The sale of Greek sovereign debt and Greek bank assets poses a significant risk of transmitting sovereign credit default swaps to the eleven Eurozone nations (Bruttin and Saure, 2015). Default risk from Greek bonds is transferred in the bond market, but this only happened at the onset of the crisis (Kohonen, 2014). According to Forbes and Rigobon (2002), they found that the contagion originated in the Hong Kong market and subsequently spread to other markets in the region and beyond.

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Another aspect worth examining is the transmission of risk and volatility between European countries. Spillover effects are observed to be substantial during both periods of economic growth and contraction, although contagion is more widespread during times of crisis (Rigobon, 2016). Any evidence of spillover or contagion weakens the Efficient Markets Hypothesis and hampers the effectiveness of risk reduction through diversification. Fry-McKibbin et al. (2017) introduce joint contagion tests that ascertain the manner in which contagion operates simultaneously through multiple channels, including co-skewness, co-kurtosis, and covolatility. By subjecting daily Eurozone equities returns from 2005 to 2014 to these tests, researchers find that contagion spreads widely through higher order moment channels during the Global Financial Crisis and the European debt crisis. This type of contagion is often missed by traditional correlation-based analyses. Alternatively, Monte Carlo simulations are employed to estimate finite sample distributions, which includes determining critical values for small sample sizes (Fry-McKibbin and Hsiao, 2016). The novel approach consistently demonstrates superior power functions compared to similar tests, allowing for early and definitive detection of contagion. The utilization of simulations in conjunction with sophisticated regression models is crucial for examining financial transmissions and contagions. In our research, we examine the contagion by studying the fat tails association between stocks and bonds using coskewness and cokurtosis.

The initial phase of the study aims to elucidate the correlation between stock market downturns and the credit risk of sovereign nations within the Eurozone. We aim to examine the transmission effect between the two markets prior to and following two major downturns: the subprime mortgage crisis and the COVID-19 pandemic. In addition, we examine the incidence of bond market crashes in 11 countries to determine if there is synchronization in terms of the occurrence of bond market shocks. Our initial step involves ascertaining the timing of stock market crashes and their potential impact on the bond market. In the second section, our objective is to examine the transmission effect between the sovereign bond market and the stock market. We will accomplish this by utilizing the Forbes-Rigobon model as well as conducting co-skewness, co-kurtosis, and covolatility tests. In order to accomplish this, we must initially ascertain instances of stock market collapses throughout our selected period. The CUSUM test will be employed for this objective. However, the pre-crisis period is critical for the detection of the magnitude and the nature of crisis. Subsequently, we assess bubble periods utilizing the SADF and R-SADF models. In addition, we study if they coincide with the pre-crisis periods giving insights for investors to predict the upcoming crisis.

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The three experiments offer indisputable proof of contagion (Muhammad et al., 2022). Rizaldi and Imam (2016) found that equities with higher betas, larger market capitalization, higher return volatility, higher debt ratios, lower levels of liquid assets, and lower asset profitability suffered larger losses on the day of collapse. Furthermore, in most instances of stock market crashes, there are immediate and lasting impacts on stock returns, both in the short-term and the long-term. The methods used to detect contagion vary based on the market's composition, microstructure, and connection to the global environment (Ming-Yuan et al., 2019). Shangmei and Junhuan (2019) examine the systemic risk of China's stock market by employing 5-minute intraday transaction data. Their research indicates that the government's response may have negatively affected the overall risk of China's stock market, as the risk increased during the 2008 financial crisis.

Crashes can trigger a stock market bubble that forms and bursts when investors collectively purchase stocks, leading to artificially inflated and unsustainable market prices. Throughout history, there has been a recurring pattern of bubbles, collapses, and financial crises. Moreover, these surges and downturns have occurred in various financial markets, encompassing established financial systems, emerging economies, and expanding financial markets. This study examines the relationship between the COVID-19 pandemic and the Global Financial Crisis, tracing back their cyclical patterns. We initiate the process by identifying the periods of crashes and their duration of continuity. Furthermore, during these time intervals, we analyze the transmission of financial market disturbances. The "big, short" phenomenon during the 2008 financial crisis is an example of how the volatility in stocks has a significant impact on other markets. On the other hand, the transition from crashes to crises is linked to a heightened focus on a particular market. Amidst the Dotcom crisis, there was an enormous surge in demand for technology stocks, resulting in the formation of a speculative bubble. Furthermore, the 2008 financial crisis was directly linked to the significant decline in the stock market and the bursting of the mortgage bubble. Hence, our study endeavors to establish a correlation between the emergence of financial market crises and the escalation of risk levels, culminating in a crisis during periods of market crashes. This paper is the first to examine the crisis cycle by utilizing the three primary components.

The COVID-19 crisis has had a notable impact on the share price bubbles of over-financialized enterprises (Wang et al., 2023), causing them to be crowded out. Additionally, it could indicate that the virtual economy is amplifying the risk of a bubble burst due to unforeseen public failures. An alternative is to explore the commodity market, where there are frequent instances of moderate volatility in both the WTI and gold markets (Gharib et al., 2021). The correlation between gold and

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oil has been thoroughly studied to determine common risks and the dynamics of their relationship. Yang and Oxley (2018) examine Japan's extensively documented asset price bubbles in the 1980s and 1990s, introducing them as subjects for contemporary econometric analyses. Their research provides compelling econometric evidence of the presence of bubbles in both Japan's stock and real estate markets during this specific time period.

Moreover, a significant contribution of their research lies in conducting formal tests to examine the transmission of bubble effects from Japan's stock market to its real estate market. In their study, Yishi et al. (2022) employ the SADF and GSADF tests to identify the existence and timing of price bubbles in the NFT and DeFi markets. Based on their research, both the NFT and DeFi markets exhibit speculative bubbles. It is worth mentioning that NFT bubbles occur more often and are of greater magnitude compared to DeFi bubbles. The price bubbles observed in the NFT and DeFi sectors exhibit significant associations with market euphoria and the prevailing uncertainty within the cryptocurrency industry. They highlight instances where no bubbles are detected, suggesting that these markets possess intrinsic value and should not be dismissed as mere speculative bubbles. We regularly utilize the GSADF and SADF in our research to detect bubbles and determine their correlation with bond market crashes.

This paper is divided into four parts. In the second part, we present the methodology used for the three models. It also describes the dataset. We then follow with the results, and finally, Section 4 concludes.

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2. Methodology and Data

2.1. Data Collection

The sample for this study comprises 11 countries and the global index for the European area, Euronext 100. In our analysis, we will employ two primary variables. The daily clean prices for the 10-year Government Benchmark Bond Indices, spanning from January 1, 2001, to December 31, 2021, can be accessed through Thomson Reuters DataStream. Additionally, in order to examine the correlation between the stock market crash and the sovereign bond market, we will utilize data obtained from the EURONEXT100 index, IBEX35 index, FTSE.MIB index, DAX index, CAC40 index, BEL20 index, FTSE.ATHEX index, AEX index, ISEQ index, OMX25 index, and ATX index. These data sets have been extracted from the Yahoo Finance Database. Table 1 provides a concise overview of the characteristics related to bonds and stocks for each country. It includes the starting and ending dates, as well as the total number of observations. The start date is 02/01/2001 using the market return calculated with the following formula $r_{t,j} = (p_{j,t} - p_{j,t-1})/p_{j,t-1}$ with $p_{j,t}$ presents the price of security j at time t, and r for return. EURONEXT100 index stands as one of the most valuable and active traded stocks in Europe. It includes a collection of the most important, highly valued, and actively traded stocks in the market, starting with 1,000 points in the year 2000.

Table 1: *Collected data description on stock and bond markets.*

Countries	Abbreviation	Start date	Last date	Stock market index	Observations
Austria	***AUS***	02/01/2001	31/12/2021	ATX	5479
Belgium	***BEL***			BEL20	
Finland	***FIN***			OMX25	
France	***FRA***			CAC40	
Germany	***GER***			DAX	
Greece	***GRE***			FTSE.ATHEX	
Ireland	***IRE***			ISEQ	
Italy	***ITA***			FTSE.MIB	
Netherlands	***NET***			AEX	
Portugal	***POR***			PSI	
Spain	***SPA***			IBEX35	
				Euronext100	5479

2.2. Cumulative Sum "CUSUM" test

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In order to identify the time periods characterized by significant fluctuations in the bond market across various countries, we employ the CUSUM method for detecting variance shocks. Sansó et al. (2004) enhance the Iterated Cumulative Sums of Squares "ICSS" process, based on the groundbreaking research by Inclan and Tiao (1994). The process entails conducting multiple iterations of the CUSUM test, which relies on Monte Carlo simulation, in order to detect points where there are significant changes in the unconditional variance. The model's main objective is to identify shifts in the variability of a specific series that are caused by external shocks. Regime shifts indicate changes in the trend following a prolonged persistence since the last detected shock. This is a statistical technique employed to identify alterations or deviations in the average value of a sequence of data over time. The CUSUM method is a valuable tool for analyzing data trends over time by calculating the cumulative differences between observed and expected future shocks in the data. The formula for CUSUM is expressed as:

$$\sigma_{xt}^2 = \begin{cases} \sigma_{x0}^2\ si\ \ 1 < t < c_1 \\ \sigma_{x1}^2\ si\ \ c_1 < t < c_2 \\ \cdots \\ \cdots \\ \cdots \\ \sigma_{xM_x}^2\ \text{si}\ \ \text{c}_{M_x} < t < T \end{cases} \#(1)$$

Where, T is the number of observations and x_t for series of independent and normally distributed observations with non-conditional variance σ_{xt}^2. M_x are the number of points indicating a regime change in the variance. C_j represents the dates of variance regime changes with $j = 1 \ldots M_x$. To estimate the number of variance regime changes, a cumulative sum of squares of residuals follows: $C_k = \sum_{t=1}^{k} u_t^2,\ k = 1 \ldots T$. Inclan and Tiao (1994) define the statistic for the detection of change with: $D_k = \left(\frac{C_k}{C_t}\right) - \left(\frac{k}{T}\right)\ with\ D_0 = D_T = 0$. In the case of no variance regime change in the series, the values of D_k oscillate around zero. However, when breakpoint points exist, D_k is strictly different from zero. Under the null hypothesis of homogeneous variance, $H_0\colon var(x_t) = \sigma^2$ (constant), the statistic D_k converges in distribution to a standard Brownian motion. The null hypothesis H_0 cannot be accepted when $K^* = max_k(\sqrt{\frac{T}{2}}|D_k|$ is outside the critical interval of ±1.358, thus K^* is a breakpoint point at the 95% threshold.

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Furthermore, the original version of the ICSS procedure is defined for homogeneous variance over an interval and does not consider the heteroscedasticity nature of financial series. Sanso et al. (2004) make a modification to the statistic D_k by considering the fourth-order moment. They replace it by $G_K = \hat{\delta}_4^{(-\frac{1}{2})}$ where $\hat{\delta}$ is defined as a consistent estimator of the fourth-order moment, expressed by $\hat{\delta} = T^{-1}\sum_{t=1}^{T}(\tau_t^2 - \hat{\sigma}^2)^2 + 2T^{-1}\sum_{t=1}^{m} w(l,m)\sum_{t=l+1}^{T}(\tau_t^2 - \hat{\sigma}^2)(\tau_{t-1}^2 - \hat{\sigma}^2)$, where $w(l,m)$ is a Bartlett window. The estimation depends on the choice of m obtained through the Newey-West method (1994). The null hypothesis H_0 is rejected when $K_s^* = max_k(\left|\frac{D_k}{\sqrt{T}}\right|$ is outside the critical interval of ±1.405. The point K_s^* is defined as a breakpoint in the variance. The ICSS algorithm detects M_j regimes of variance for each series j (j ranging from 1 to N). Let S_{jt}^i be indicator variables of the existence of variance regime changes $with\ i = 1 \dots M_j$. S_{jt}^i takes the value 1 in the interval] C_i, C_{i+1} [between the points i and $i+1$ of trend change and takes the value zero outside of this interval.

2.3. Contagion effect: Forbes and Rigobon, co-skewness, co-kurtosis and Covolatility tests

The primary objective of the Forbes-Rigobon model is to assess the degree of co-movements or correlations among various stock markets, rather than explicitly examining the presence of contagion. During times of financial turmoil, there is an increased level of time-varying interconnectedness among interrelated markets. The model calculates the degree of simultaneous movement between pairs of analyzed markets using correlation or covariance measures. The method relies on a dynamic correlation approach to accurately capture fluctuations in the relationships between variables over time. Subsequently, it establishes threshold levels to differentiate between typical market connections and periods of heightened simultaneous movements. The threshold is contingent upon the pre-established statistical criteria. The contagion test developed by Forbes and Rigobon (2002) is characterized by an elevation in the heteroskedastic-adjusted correlations between periods of crisis (bust) and non-crisis periods (boom), as expressed by the following equation:

$$v_{y|x_i} = \frac{\rho_y}{\sqrt{1 + \frac{\sigma_{yi}^2 - \sigma_{xi}^2}{\sigma_{xi}^2}(1 - \rho_y^2)}}$$

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With x is the non-crisis = boom period. And y is the crisis or pandemic period = bust period. ρ_y is the bust period correlation. σ_{yi}^2, σ_{xi}^2 are respectively the bust and the boom period volatility for the market of origin. $\frac{\sigma_{yi}^2-\sigma_{xi}^2}{\sigma_{xi}^2}$ represents the proportionate change in the variance. If there is no contagion from market i to market j, the null hypothesis $H_0: \rho_x = \rho_y$ is verified. Otherwise, there is contagion between markets i and j. The calculation of the t-statistic using Forbes and Rigobon (2002) model is as follow:

$$FR\ (i \to j) = \frac{\frac{1}{2}\ln\left(\frac{1+v_y}{1-v_y}\right) - \frac{1}{2}\ln\left(\frac{1+\rho_x}{1-\rho_x}\right)}{\sqrt{\frac{1}{T_y-3}+\frac{1}{T_x-3}}}$$

Where T_x and T_y are the sub-sample size of boom-and-bust period, respectively.

The contagion test developed by Fry, Martin, and Tang (2010) involves examining alterations in co-skewness between boom period and bust period. They examine the contagion by calculating the stocks return squared and bond return and vice versa. The co-skewness test represents another alternative to find out the correlation in the tails between markets. It is an indicator of the existence of correlated heavy tails in time of bust or crisis. The formula for assessing the co-skewness is represented as follows:

$$\psi(1,2) = \frac{1}{T}\sum^{T} \frac{r_{it}-\mu_i}{} \left(\frac{r_{jt}-\mu_j}{}\right)^2$$

$$\psi(2,1) = \frac{1}{T}\sum^{T} \left(\frac{r_{it}-\mu_i}{}\right)^2 \frac{r_{jt}-\mu_j}{}$$

Where T stands for the sub-samples' size. With r_i and r_j are the stock return and bond return respectively in both periods. Moreover, μ_i and μ_j are, respectively, the stock and bond mean. σ_i and σ_j are respectively, the volatility of stock and bond market period according to Forbes and Rigobon (2002). To evaluate the null hypothesis Fry, Martin, and Tang (2010) propose the following model using the Chi-square:

$$\chi^2 = \left(\frac{\psi_y - \psi_x}{\sqrt{\frac{4v_y^2+2}{T} + \frac{4\rho_x^2+2}{T}}}\right)^2$$

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Where $\psi_y - \psi_x$ is the difference between coskewness (based on $\psi(1,2)$ and $\psi(2,1)$) in boom and in bust period. ν_y is the adjusted correlation for heteroscedasticity according to Forbes and Rigobon (2002). ρ_x is the bust period correlation. Finally, T_x and T_y are the sub-sample size.

Based on the works of Fry, Martin, and Tang (2010) and Forbes and Rigobon (2002), Fry-McKibbin and Hsiao (2018) suggest that contagion can be detected by a change in co-kurtosis. They examine the contagion using the first variable's mean and the second variable' skewness. The formula is as follows:

$$\xi(1,3) = \frac{1}{T}\sum_{t=1}^{T} \frac{r_{it} - \mu_i}{\sigma_i}\left(\frac{r_{jt} - \mu_j}{\sigma_j}\right)^3$$

$$\xi(3,1) = \frac{1}{T}\sum_{t=1}^{T} \left(\frac{r_{it} - \mu_i}{\sigma_i}\right)^3 \frac{r_{jt} - \mu_j}{\sigma_j}$$

To assess the null hypothesis, Fry-McKibbin and Hsiao (2018) build a statistic test using the chi-square, which is calculated using the following formula:

$$\chi^2 = \left(\frac{\xi_y - \xi_x}{\sqrt{\frac{18\nu_y^2 + 6}{T} + \frac{18\rho_x^2 + 6}{T}}}\right)^2$$

Where $\xi_y - \xi_x$ is the difference between cokurtosis (based on $\xi(1,3)$ and $\xi(3,1)$) for the boom-and-bust periods.

Additionally, Fry-McKibbin and Hsiao (2018) propose the covolatility test which is calculated using the dependencies in the volatility across the studied markets. The variance is the essential of measure and testing for the covolatility model represented with the following formula:

$$\xi(2,2) = \frac{1}{T}\sum_{t=1}^{T} \left(\frac{r_{it} - \mu_i}{\sigma_i}\right)^2 \left(\frac{r_{jt} - \mu_j}{\sigma_j}\right)^2 - (1$$

For the test of the null hypothesis, the chi-squared test is calculated as follows:

$$\chi^2 = \left(\frac{\xi_y - \xi_x}{\sqrt{\frac{4\nu_y^4 + 16\nu_y^2 + 4}{T} + \frac{4\rho_x^4 + 16\rho_x^2 + 4}{T}}}\right)^2$$

Where $\xi_y - \xi_x$ is the difference between covolatility in boom and in bust period. According to Muhammad (2022), each of the tests for co-skewness, co-kurtosis, and covolatility involves a

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statistic that follows a chi-square distribution, moving from boom to bust period. During this transition, the co-skewness test compares the mean return of asset i related to the volatility of asset j. Furthermore, the co-kurtosis measures the relationship between the anticipated return of asset i and the skewness of asset j. For the covolatility test, it assesses the variances of the returns of the two assets. Top of the form Bubbles: The Supremum Augmented Dickey-Fuller (SADF) and Generalized Supremum Augmented Dickey-Fuller (R-SADF) models

2.4. SADF Model: Phillips, Yu, and Wu (2011)

The SADF model, developed by Phillips et al. (2011), was designed to identify and quantify the existence of bubbles in time series data, with a specific focus on financial markets. The mentioned test is an expansion of the Augmented Dickey-Fuller (ADF) test, a widely utilized method in econometrics for conducting unit root tests. A unit root indicates that a time-series is non-stationary and has a random trend, which is analogous to the notion of bubbles in financial markets. The critical values for the SADF test are derived via Monte Carlo simulations. The statistical distribution of the data is non-standard and varies depending on the number of potential break points being considered. The SADF test entails the computation of the ADF test statistic at every conceivable break point. The SADF test chooses the highest value as the critical value.

The G-SADF model, proposed by Phillips et al. (2015), is an extension of the SADF model that applies it recursively using a rolling window of data. This approach is beneficial because financial market bubbles are typically temporary and transitory, manifesting and dissipating over time. At each iteration, the latest observation is added to the window, while the oldest one is removed. This process persists throughout the entire time series, producing a sequence of SADF statistics. The G-SADF model utilizes a rolling window to detect the existence of bubbles at various time intervals, thereby offering valuable insights into the progression of bubble-like phenomena. Furthermore, Homm and Breitung (2012) have shown through Monte Carlo simulations that the SADF test has a higher level of statistical power in identifying individual price bubbles that collapse periodically.

The SADF test is based on iteratively estimating the standard ADF regression with a fixed starting point (P_0) and with expanding window (P_w). The window size (P_w) varies from P_0 to 1, and the starting point (P_1) is always fixed at 0, making the endpoint of each P_2 equal to P_w. Consequently, the ADF regression is repeatedly computed while incrementally increasing the window size (P_2), ranging from P_0 to 1. At each step of the recursive estimation process from 0 to P_2, an ADF statistic is generated, denoted as $ADF_0^{P_2}$. The SADF test is then defined as the

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supremum statistic of the $ADF_0^{P_2}$ sequence for $P_2 \in [P_2, 1]$, derived from the forward recursive regression as follow:

$$SADF(P_0) = \sup_{P_2 \in [P_0, 1]} ADF_0^{p_2}$$

The ADF regression is denoted as $\Delta y_t = \alpha + \beta_T + \gamma y_{t-1} + \delta_1 \Delta y_{\,t-1} + \cdots + \delta_{m-1} \Delta y_{t-m+1} + \varepsilon_t$. The null hypothesis $H_0: \gamma = 1$ represents a unit root, and $H_1: \gamma > 1$ denotes mildly explosive process. α is a constant, β and γ are the coefficients for the lagged variable. β_T is the sum of a deterministic trend, and δ is the theoretical autocorrelation, with a lag order m and the stationary error process ε_t. Phillips et al. (2015) show that the key benefit of the GSADF test resides in its ability to accommodate more flexible estimation windows. This means that the starting point (P_1) is permitted to vary within the range of 0 to $(P_2 - P_0)$. The GSADF test can be mathematically represented as follows:

$$GSADF(P_0) = \sup_{P_1 \in [0, P_2 - P_0], P_2 \in [P_0, 1]} ADF_{P_1}^{p_2}$$

As demonstrated by Phillips et al. (2011) and Phillips et al. (2015), the date-stamping method can be effectively applied to both the SADF and GSADF tests, ensuring consistent estimation of beginning and end of bubbles. In the date-stamping SADF test, financial price data is sorted chronologically to form a time-series dataset. To measure a bubble price initiating at time T_{P2}, each element of the estimated $ADF_0^{P_2}$ sequence is compared to the corresponding right-tailed critical values of the standard ADF statistic. The estimated start points of a price bubble, denoted as T_{Ps}, is identified as the $ADF_0^{P_2}$ value that crosses the corresponding critical value from below. On the other hand, the estimated endpoint of a price bubble, denoted as T_{Pe}, is determined as the $ADF_0^{P_2}$ value that crosses the critical value. The formula for the determination of the bubbles' dates is as follows:

$$\hat{P}_s = \inf_{P_2 \in [P_0, 1]} \left\{ P_2 : ADF_0^{P_2} > cv_{P_2}^{\beta_T} \right\},$$

$$\hat{P}_e = \inf_{P_2 \in [\hat{P}_s, 1]} \left\{ P_2 : ADF_0^{P_2} < cv_{P_2}^{\beta_T} \right\},$$

Where T and β_T are close to 0. $cv_{P_2}^{\beta_T}$ represents the $100(1 - \beta T)\%$ critical value of the standard ADF statistic based on T_{P2} observations. Furthermore, the price bubble period based on the date stamping GSADF test will be estimated as follow.

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$$\hat{P}_s = \inf_{P_2 \in [c_0, 1]} \left\{ P_2 : BSADF_0^{P_2}(P_0) > cv_{p_2}^{\beta_{T_{P_2}}} \right\},$$
$$\hat{P}_e = \inf_{P_2 \in [\hat{c}_s, 1]} \left\{ P_2 : BSADF_0^{P_2}(P_0) < cv_{p_2}^{\beta_{T_{P_2}}} \right\},$$

Where T and β_T are close to 0. $cv_{p_2}^{\beta_{T_{P_2}}}$ represents the $100(1 - \beta T\ )\%$ critical value of the standard ADF statistic based on T_{P2} observations. The backward sup ADF statistic is $BSADF(P_0)$ for $P_2 \in\ [P_0, 1]$. Therefore, to associate BSADF(P_0) to GSADF statistic, we can use:

$$GSADF(\gamma_0) = \sup_{P_2 \in\ [P_0, 1]} BSADF_0^{P_2}(P_0)$$

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