Computation of Stabilizing PID Controllers for Time Delay Systems

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***Abstract*— In this paper, we give a new method for computing the set of all stabilizing PID controllers applied to a strictly proper linear system with time delay. First, a necessary condition and a constant gain stabilizing algorithm are used to calculate the admissible ranges of one of the controller's parameters. Then, for a fixed value of this parameter the stabilizing regions in the remaining two parameters are determined using the D-decomposition method. Finally, an illustrative example is given to show the effectiveness of the proposed approach.**

***Keywords*— PID controller, Time delay systems, Stability, D-decomposition, stability, Hermite-Biehler theorem.**

I. Introduction

Recently, several computational methods have been proposed to determine the set of all stabilizing first-order controllers for delay free linear time-invariant systems [1], [2], [3], [4]. In fact, the quest for an analytic design method for low-order controllers, being phase-lead or phase-lag or PID has been around for decades. The main motivation for this need comes from the fact that these controllers are widely used in industry. In [5] it is reported that more than 90% of the controllers used in industry are PID controllers. Moreover, it is easier to tune these three controllers and control high-order plants by low-order controller.

Many control systems are represented by plants with time delay, that is why stability of time delay systems is still an active area of research. Recently, there has been a great interest in computing stabilizing regions in the parameter space of simple controllers. Using extensions of Hermite-Biehler theorem the set of all stabilizing PID controllers are determined for first-order systems with dead time in [6]. A different approach was used in [7] to determine the stability region in the parameter space of PID-controlled second order systems with time delay. The D-decomposition method was used in [8] to get the stabilizing regions of a PID controller and in [9] and [10] to get the stabilizing regions of a first-order controller for an all poles systems with delay. In this paper, the stabilizing regions in the parameter space of a PID controller for a strictly proper system with delay are determined.

The paper is organized as follows. In section II, a constant gain stabilizing algorithm is used to determine the admissible ranges of one of the controller's parameters. Next, the stabilizing regions in the space of the remaining two parameters are determined using the D-decomposition method. In section III, an illustrative example is given. Section IV contains some concluding remarks.

II. Stabilizing Proportional Controllers

To begin with, let us give the following preliminary results from [2], applicable to rational transfer functions without delay. These results will be applied to get an estimate of the admissible stabilizing ranges of one of the controller's parameters. Let us first fix the notation used in this paper. Let **R** denote the set of real numbers and C denote the set of complex numbers and let C_- , C_0 , C_+ denote the points in the open left-half, $j\omega$-axis, and the open right-half of the complex plane, respectively. Given a set of polynomials $\psi_1,\dots,\psi_l \in R[s]$ not all zero and l > 1, their *greatest common divisor* is unique and it is denoted by *gcd* $\{\psi_1,\dots,\psi_l\}$. If *gcd* $\{\psi_1,\dots,\psi_l\}$ = 1, then we say $(\psi_1,\dots,\psi_l)$ is *coprime*. The derivative of ψ is denoted by ψ'. The set h of Hurwitz stable polynomials are

$$h=\{\psi(s)\in R[s]:\psi(s)=0\Rightarrow s\in C_-\}$$

The *signature* $\sigma(\psi)$ *of a polynomial* $\psi(s)\in R[s]$ is the difference between the number of its C_- roots and C_+ roots. Given $\psi(s)\in R[s]$, *the even-odd components* (a,b) *of* $\psi(s)$ are the unique polynomials $a,b \in R[s^2]$, such that

ISSN:2356-5608

$$\psi(s) = a(s^2) + sb(s^2)$$

It is possible to state a necessary and sufficient condition for the Hurwitz stability of $\psi(s)$ in terms of its even-odd components (a,b). Stability is characterized by the interlacing property of the real, negative, and distinct roots of the even and odd parts. This result is known as the Hermite-Biehler theorem. Below is a generalization of the Hermite-Biehler theorem applicable to not necessarily Hurwitz stable polynomials. Let us define the *signum function* $\xi: R \to \{-1,0,1\}$ by

$$\xi u = \begin{cases} -1 & \text{if } u < 0 \\ 0 & \text{if } u = 0 \\ 1 & \text{if } u > 0 \end{cases}$$

Lemma 1. [12] *Let a nonzero polynomial* $\psi \in \mathbf{R}[s]$ *have the even-odd components* (a,b). *Suppose* $b \neq 0$ *and* (a,b) *is coprime. Then,* $\sigma(\psi) = r$ *if and only if at the real negative roots of odd multiplicities* $v_1 > v_2 > \ldots v_l$ *of b the following holds:*

$$r = \begin{cases} \xi b(0)\begin{pmatrix} \xi a(0) - 2\xi a(v_1) + 2\xi a(v_2) + \\ \ldots + (-1)^l 2\xi a(v_l) \end{pmatrix} & \text{if } \deg\psi \text{ odd} \\ \xi b(0)\begin{pmatrix} \xi a(0) - 2\xi a(v_1) + 2\xi a(v2) + \\ \ldots + (-1)^{l+1} 2\xi a(-\infty) \end{pmatrix} & \text{if } \deg\psi \text{ even} \end{cases}$$

The following result, determines the number of real negative roots of a real polynomial.

Lemma 2. [2] *A nonzero polynomial* $\psi \in \mathbf{R}[s]$, *such that* $\psi(0) \neq 0$, *has r real negative roots without counting the multiplicities if and only if the signature of the polynomial* $\psi(s^2) + s\psi'(s^2)$ *is 2r. All roots of* ψ *are real, negative, and distinct if and only if* $\psi(s^2) + s\psi'(s^2) \in h$.

We now describe a slight extension of the constant stabilizing gain algorithm of [12]. Given a plant

$$G(s) = \frac{p(s)}{q(s)}$$

where $p,q \in \mathbf{R}[s]$ are coprime with m = deg p less than or equal to n = deg q, the set

$$\phi_r(p,q) := \{\alpha \in R, \sigma[\phi(s,\alpha)] = \sigma[q(s) + \alpha p(s)] = r\}$$

is the set of all real α such that $\phi(s,\alpha)$ has signature equal to r.

Let (h,g) and (f,e) be the even-odd components of $q(s)$ and $p(s)$, respectively, so that

$$q(s) = h(s^2) + sg(s^2)$$
$$p(s) = f(s^2) + se(s^2)$$

Let (H,G) be the even-odd components of $q(s)p(-s)$. Also let $F(s^2) := p(s)p(-s)$. By a simple computation, it follows that (s^2 is replaced by u):

$$\begin{aligned} H(u) &= h(u)f(u) - ug(u)e(u) \\ G(u) &= g(u)f(u) - h(u)e(u) \\ F(u) &= f^2(u) - ue^2(u) \end{aligned} \tag{1}$$

With this setting, we have

$$[q(s) + \alpha p(s)]p(-s) = [H(s^2) + \alpha F(s^2)] + sG(s^2)$$

If $G \neq 0$ and if they exist, let the *real negative roots with odd multiplicities of* $G(u)$ be $\{v_1,\ldots,v_l\}$ with the ordering $v_1 > v_2 > \ldots v_l$, with $v_0 := 0$ and $v_{l+1} := -\infty$ for notational convenience.

The following algorithm determines whether $\Phi_r(p,q)$ is empty or not and outputs its elements when it is not empty [9]:

Algorithm 1.

1) *Consider all the sequences of signums*

$$D = \begin{cases} \{i_0, i_1, \ldots, i_l\} & \text{for odd } r - m \\ \{i_0, i_1, \ldots, i_{l+1}\} & \text{for even } r - m \end{cases}$$

where $i_j \in \{-1,1\}$ for j=0,1,…,l+1

2) *Choose all the sequences that satisfy*

$$r - \sigma(p) = \begin{cases} i_0 - 2i_1 + 2i_2 - 2i_3 + \ldots + 2(-1)^l i_l & \text{for odd } r - m \\ i_0 - 2i_1 + 2i_2 - 2i_3 + \ldots + 2(-1)^{l+1} i_{l+1} & \text{for even } r - m \end{cases}$$

3) *For each sequence of signums* $D = \{i_j\}$ *that satisfy step 2, let*

ISSN:2356-5608

$$\alpha_{max} = \max\left\{-\frac{H}{F}(v_j)\right\} \quad \forall v_j \text{ for wich } F(v_j) \neq 0 \text{ and } i_j \xi F(v_j) = 1$$

and

$$\alpha_{min} = \min\left\{-\frac{H}{F}(v_j)\right\} \quad \forall v_j \text{ for wich } F(v_j) \neq 0 \text{ and } i_j \xi F(v_j) = -1$$

The set $\Phi_r(p,q)$ is non-empty if and only if for at least one signum sequence D satisfying step 2, $\alpha_{max} < \alpha_{min}$ holds.

4) *$\Phi_r(p,q)$ is equal to the union of intervals $(\alpha_{max}, \alpha_{min})$ for each sequence of signums D that satisfy step 3.*

The algorithm above is easily specialized to determine all stabilizing proportional controllers $C(s) = \alpha$ for the plant $G(s)$. This is achieved by replacing r in step 3 of the algorithm by n, the degree of $\phi(s,\alpha)$.

Remark 1. By Step 3 of Algorithm 1, a necessary condition for the existence of a $\alpha \in \Phi_r(p,q)$ is that the odd part of

$$[q(s) + \alpha p(s)]p(-s)$$

has at least $\underline{r} = \left\lfloor \frac{|-\sigma(p)|-1}{2} \right\rfloor$ real negative roots with odd multiplicities. When solving a constant stabilization problem, this lower bound is $\underline{r} = \left\lfloor \frac{n-\sigma(p)-1}{2} \right\rfloor$.

III. STABILIZING PID CONTROLLERS

In this section we consider determining the stabilizing regions in the parameter space of a PID controller.

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s}$$

applied to a strictly proper system with time delay

$$G(s) = \frac{b_m s^m + \ldots + b_0}{a_n s^n + \ldots + a_0} e^{-Ls} = \frac{p(s)}{q(s)} e^{-Ls}$$

where $L > 0$ is the time delay and $deg(q) = n > m = deg(p)$. First the admissible ranges of the first parameter k_p are found. Then, k_p is fixed within this range and the D-decomposition method is used to determine the stabilizing regions in the plane of the remaining two parameters k_i and k_d. By sweeping over the values of k_p the complete set of stabilizing controllers can be obtained. The exact range of stabilizing k_p values is difficult to determine analytically. In fact for the similar problem of finding the stabilizing regions of a PID controller, determining the exact range of stabilizing k_p values analytically is solved for the only case of a first-order plant with time delay [6]. Therefore instead of determining the exact range of k_p, a necessary condition will be used to get an estimate of the stabilizing range of k_p. Alternatively, we can fix k_d and determine the stabilizing regions in the plane of (k_p, k_i) as will be shown in the next sub-section.

A. Determining the admissible range of one parameter

Consider a PID controller

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s}$$

applied to a plant transfer function

$$G(s) = \frac{p(s)}{q(s)} e^{-Ls}$$

In this part our aim is to find all admissible values of k_p. Replacing the time delay by a Padé approximation of order l

$$e^{-Ls} \approx \frac{r(-s)}{r(s)}$$

where $r(s) = h$, we get the following closed-loop characteristic polynomial

$$\phi_0(s, k_p, k_i, k_d) = sq(s)r(s) + (k_d s^2 + k_p s + k_i)p(s)r(-s)$$
$$= sq_0(s) + (k_d s^2 + k_p s + k_i)p_0(s)$$

where

$$q_0(s) = q(s)r(s)$$
$$p_0(s) = p(s)r(-s)$$

Multiplying $\phi_0(s, k_p, k_i, k_d)$ by $p_0(-s)$, we obtain

ISSN:2356-5608

$$\psi_0(s,k_p,k_i,k_d) = \phi_0(s,k_p,k_i,k_d)p_0(-s) = s^2G(s^2) + k_d sF(s^2) + k_iF(s^2) + s\left[H(s^2) + k_pF(s^2)\right],$$

where H, G and F are given by (1). Note that the even part of $\psi_0(s)$ has only two parameters (k_d, k_i) and the odd part of $\psi_0(s)$ has only one parameter k_p. By Remark 1 the odd part $H(u) + k_pG(u)F(u)$ of ψ_0 must have at least $r_1 = \left\lfloor \frac{n+1-\sigma(p_0)}{2} \right\rfloor$ real, negative, distinct roots. At this step two parameters (k_d, k_i) are eliminated and an auxiliary problem with only one parameter will be solved. Let

$$\phi_1(s,k_p) = \left[H(s^2) + sH'(s^2)\right] + k_p\left[F(s^2) + sF'(s^2)\right]$$

using Lemma 2, finding values of k_p such that $H(u) + k_pF(u)$ has r_1 real, negative, distinct roots is equivalent to finding values of k_p such that the new constructed polynomial $\phi_1(s,k_p)$ has signature equals to $2r_1$. Alternatively, we can use the even part of $\psi_0(s,k_p,k_i,k_d)$ and repeat the same reasoning as above to determine admissible values of k_d or k_i. Note that the even part of $\psi_0(s,k_p,k_i,k_d)$ has only two parameters k_d and k_i, hence the D-decomposition method can be used to get the admissible values of k_d or k_i. Using Lemma 2, finding values of (k_d,k_p) such that $s^2G(s^2) + k_dsF(s^2) + k_iF(s^2)$ has r_2 real, negative roots is equivalent to calculation values of (k_d,k_p) such that

$$\phi_2(s,k_d,k_i) = \left[G_1(s^2) + sG_1'(s^2)\right] + k_d\left[F_1(s^2) + sF_1'(s^2)\right] + k_i\left[F(s^2) + sF'(s^2)\right]$$

has signature equals to $2r_2$ which can be solved using D-decomposition method, where $G_1(s^2) = s^2G(s^2)$ and $F_1(s^2) = s^2F(s^2)$.

B. Stabilizing regions of PID controllers

In this sub-section, we choose to fix k_d within the range determined by the above procedure, and use the D-decomposition method to determine the stabilizing regions in the plane of the remaining two parameters k_p and k_i. The reason of this choice is to treat PI controllers (proportional integral) as a special case for k_d=0.

The closed loop characteristic function is given by

$$\phi(s) = sq(s) + (k_ds^2 + k_ps + k_i)p(s)e^{-Ls} \qquad (2)$$

The D-decomposition method is based on the fact that roots of the quasi-polynomial (2) change continuously when the coefficients are changed continuously. Hence, a stable quasi-polynomial can become unstable if and only if at least one of its roots crosses the imaginary axis. Using this fact, the plane of (k_p,k_i) can be partitioned into regions with the same number of roots of (2) in the left-half plane. Stability can be checked by choosing a point inside a region and applying classical methods such as the Nyquist criterion. Evaluating the characteristic function at the imaginary axis is equivalent to replacing s by $j\omega$, $\omega \geq 0$ in (2), we get

$$\begin{aligned}\phi(j\omega) = & -\omega I_q(\omega)\cos(L\omega) - \omega R_q(\omega)\sin(L\omega) + (k_i - \omega^2 k_d)R_p(\omega) - \omega k_pI_p(\omega) \\ & + j\left[\begin{aligned}&\omega R_q(\omega)\cos(L\omega) - \omega I_q(\omega)\sin(L\omega) \\ &+ \omega k_pR_p(\omega) + (k_i - \omega^2k_d)I_p(\omega)\end{aligned}\right]\end{aligned} \qquad (3)$$

where $q(j\omega) = R_q(\omega) + jI_q(\omega)$ and $p(j\omega) = R_p(\omega) + jI_p(\omega)$

Three cases will be investigated:

•**Case 1.** Setting $\omega = 0$ corresponds to the case of a root crossing the imaginary axis through the real line. This leads to the following equation:

$$k_ip(0) = 0$$

•**Case 2.** When $\omega \to \infty$ corresponds to a root leaving the left-half plane (alternatively the right half-plane) at infinity. Since e^{Ls} does not have any finite roots, we consider the quasipolynomial:

$$\phi^*(s) = \phi(s)e^{Ls} = sq(s)e^{Ls} + (k_ds^2 + k_ps + k_i)p(s)$$

which has the principal term [11].

Clearly the quasipolynomial $\phi^*(s)$ possesses a root chain of retarded type that goes deep in the left-hand plane and does not affect stability properties [11].

•**Case 3.** By sweeping over all $\omega > 0$, we consider the case of a pair of conjugate complex roots crossing the imaginary axis. Setting the real and imaginary parts of

ISSN:2356-5608

equation (3) to zero we get the pair of equations expressed in matrix format by equation (4).
Determinant of the matrix at the left-hand of (4) is equal to

$\det(\omega) = -\omega(R_p^2(\omega) + I_p^2(\omega))$

Hence, the pair (k_p, k_i) can be determined for a fixed value of k_d as given by equation (5). By sweeping over values of $\omega > 0$, the plane of (k_p, k_i) can be partitioned into regions with the same number of roots of (2) in the left-half plane.

$$\begin{bmatrix} -\omega I_p(\omega) & R_p(\omega) \\ \omega R_p(\omega) & I_p(\omega) \end{bmatrix} \begin{bmatrix} k_p \\ k_i \end{bmatrix} = \begin{bmatrix} \omega(I_q(\omega)\cos(L\omega) + R_q(\omega)\sin(L\omega) - \omega^2 k_d R_p(\omega) \\ \omega(I_q(\omega)\sin(L\omega) - R_q(\omega)\cos(L\omega) + \omega^2 k_d I_p(\omega) \end{bmatrix} \quad (4)$$

$$\begin{cases} k_p = \dfrac{1}{\det(\omega)}\left[\left(\omega(I_q(\omega)\cos(L\omega)+R_q(\omega)\sin(L\omega)) + \omega^2 k_d R_p(\omega)\right) I_p(\omega) - \left(\omega(I_q(\omega)\sin(L\omega)-R_q(\omega)\cos(L\omega)) + \omega^2 k_d I_p(\omega)\right) R_p(\omega)\right] \\ k_i = \dfrac{1}{R_p^2(\omega)+R_p^2(\omega)}\left[\left(\omega(I_q(\omega)\cos(L\omega)+R_q(\omega)\sin(L\omega)) + \omega^2 k_d R_p(\omega)\right) R_p(\omega) + \left(\omega(I_q(\omega)\sin(L\omega)-R_q(\omega)\cos(L\omega)) + \omega^2 k_d R_p(\omega)\right) I_p(\omega)\right] \end{cases} \quad (5)$$

IV. ILLUSTRATIVE EXAMPLE

Consider stabilizing the third order plant, with a right-half plane zero, given by

$$G(s) = \frac{s-3}{s^3 + 2s^2 + 3s + 5} e^{-0.25s}$$

by a PID controller. First the time delay is replaced by a first order Padé approximation and the admissible ranges of stabilizing k_d values are calculated. The auxiliary problem that we solve at this step is stabilizing the new constructed polynomial given by:

$$\begin{aligned} \phi_2(s, k_d, k_i) = {} & (21s^6 + 63s^5 + 478s^4 + 956s^3 + 1136s^2 + 1136s) \\ & + k_d(s^6 + 3s^5 - 73s^4 - 146s^3 + 576s^2 + 576s) \\ & + k_i(s^4 + 2s^3 - 73s^2 - 73s + 576) \end{aligned}$$

The solution of this auxiliary problem can be achieved by using the D-decomposition method. Fig. 1 shows the stability region in the (k_d, k_i) plane.

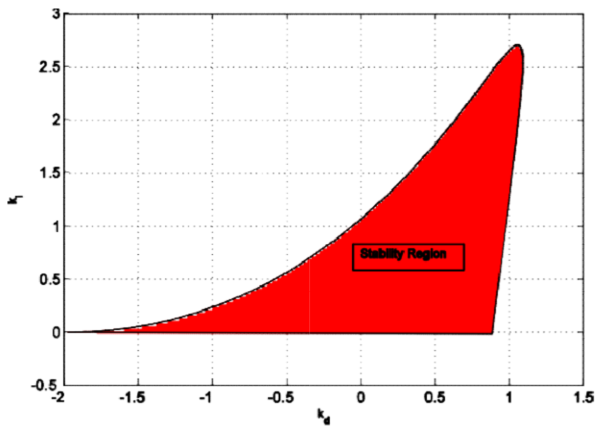

Fig. 1 Stability region in the (k_d, k_i) plane for the auxiliary problem

Now let us fix k_d within the admissible range and go back to the original problem. Let $k_d = 0$ (case of PI controller), using the method described in section III.B we get the stabilizing region in (k_p, k_i) plane as shown in Fig. 2 .

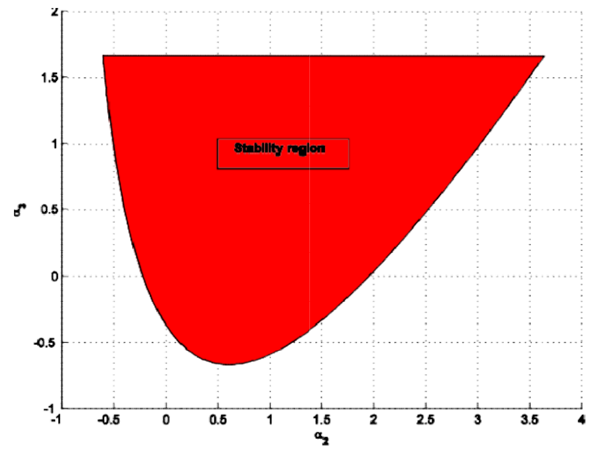

Fig. 1 Stability region of a PI controller

ISSN:2356-5608

Finally, Fig. 3 shows a 3D plot of the stabilizing regions of PID controller.

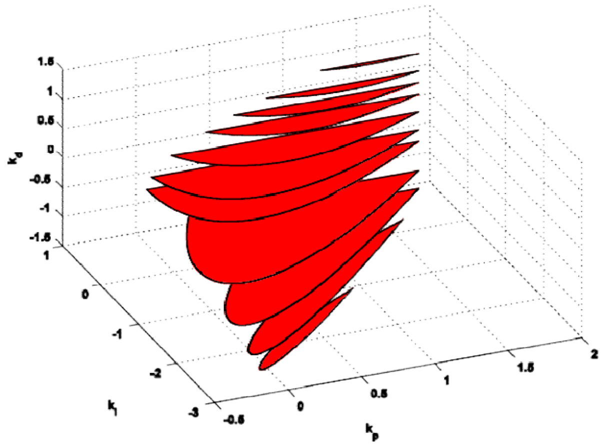

Fig. 3 A 3D plot of the stability regions

V. CONCLUSIONS

In this paper, stabilizing PID controllers are determined for a strictly proper system with time delay. The D- decomposition method is used for determining the stabilizing region in the plane of (k_p , k_i) of a PID controller. The third parameter k_d is fixed a priori within an admissible range determined using a necessary condition. These results can be extended to include gain and phase margin specifications. Moreover, once the stabilizing regions are found several performance specifications, such as maximum percent overshoot, rise time and settling time, can be evaluated and optimal PID controllers can be determined.

REFERENCES

[1] K. Saadaoui and A. B. Ozguler, "Fixed order controller design: a parametric approach", Lambert academic publishing, ISBN 987-3-8383-02126, Germany, 2010.

[2] K. Saadaoui, Fixed order controller design via parametric methods, Ph. D. Thesis, Bilkent University, 2003.

[3] R. N. Tantaris, L.H. Keel and S. P. Bhattacharyya,"Stabilization of discrete-time systems by first-order controllers", IEEE Trans. Automat. Cont., vol. 48, pp. 858-860, 2003.

[4] K. Saadaoui and A. B. Ozguler, "Stabilizing first-order controllers with desired stability region", Control and Intelligent systems, vol. 37, pp. 31-38, 2005.

[5] A. Datta, M. T. Ho and S. P. Bhattacharyya, "Structure and synthesis of PID controllers", Springer-Verlag, London 2000.

[6] G. J. Silva, A. Datta and S. P. Bhattacharyya,"New results on the synthesis of PID controllers", IEEE trans. on Automat. Cont., vol. 47, pp. 241-252, 2002.

[7] D. Wang, "Further results on the synthesis of PID controllers", IEEE Trans. Automat. Cont., vol. 52, pp. 1127-1132, 2007.

[8] K. Saadaoui, S. Elmadssia and M. Benrejeb,"Stabilizing PID controllers for a class of time delay systems", In PID controller design approaches theory, tuning and applications to frontier areas, ISBN 978-953-51-0405- 6, edited by Marialena Vagia, Intech publishing 2012.

[9] S. Testouri, K. Saadaoui and M. Benrejeb," Analytical design of first-order controllers for the TCP/AQM systems with time delay", International Journal of Information Technology, Control and Automation, vol. 2, pp. 27-37, 2012.

[10] S. Testouri, K. Saadaoui and M. Benrejeb," Stabilizing first-order controllers for TCP/AQM networks", Computer Technology and Application, vol. 2, pp. 979-984, 2012.

[11] N. Hohenbicher and J. Ackermann, "Computing stable regions in parameter spaces for a class of quasipolynomials", in Proceedings of 4th IFAC workshop on time delay systems TDS'03, France, 2003.

[12] A. B. Ozguler and A. A. Koçan, "An analytic determination of stabilizing feedback gains", Report, Institut f ̈ur Dynamische Systeme, Report no. 321, Universitat Bremen, 1994.

ISSN:2356-5608