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Diffusive systems Modeling

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***Abstract*— This paper deals with the modeling of the diffusive systems using fractional order transfer functions. We present two models able to modeling diffusive behavior. The first one is based on temporal approach and the second one is obtained using a non-integer integrator operator where the fractional behavior acts only over a limited frequency band. The diffusive system to be modeled is a thermal test bench of an aluminum heated bar at one extremity. A comparative study was presented to show the practically usefulness of the two proposed models and comparative remarks are given.**

Keywords— modeling, non-integer derivation, fractional integration operator, diffusion equation, heat conduction, thermal test bench.

I. INTRODUCTION

Non-integer order systems, that characterize diffusive interface, have been introduced long ago in various fields of science, like physics, mechanic, electricity, chemistry, biology, economy [1,2,3 and 4]. A well-known example is the case of heat transfer [5,6 and 7], such system obeys to diffusion phenomenon where the flux and the temperature are interrelated through non-integer order operators.

The objective of this paper is to analyse the behaviour of a diffusive interface, and develop tow fractional order models describing heat transfer phenomena through the aluminum bar using this analysis. The first model is based on temporal approach and the second one relies on frational operator proposed by Trigeassou [8]. Then, proposed models will be compared to the real response thanks to an experimental thermal system to show the practically usefulness of the two proposed models. Comparative discussion was also elaborated.

This paper is organized as follows: in Section 2, the experimental thermal system is presented briefly; third section is dedicated to introduce the fundamental equations governing the heat diffusion and we will show that when the heating temperature generated by a heat flux source, we can define fractional impedance. Section 4 provides a two way to modeling heat transfer the first one is dependent on temporal approach and the second is relied on fractional integrator introduced par Trigeassou [8]. In section 5, proposed models are finally tested on an experimental thermal system and we conclude this paper with and some comparison remarks.

II. EXPERIMENTAL THERMAL SYSTEM

The thermal test bench consists, as shown in Fig.1, of 1cm radius and 41cm length cylindrical bar of aluminum, under a heating resistance thermally isolated with foam that ensures a unidirectional transfer of the heating flow. The thermal system is considered as a semi-infinite dimension due to its important length compared with its section.

The input signal of the system is a thermal flow, generated from a heating resistance fixed at the extremity of the bar, commanded by a computer and the output is the temperature of the bar measured at a distance *d* of the heated surface. The thermal flow is controlled by a computer equipped with a PCI-DAS1002 input-output card.

The bar temperature is measured with the temperature sensor LM35DZ.

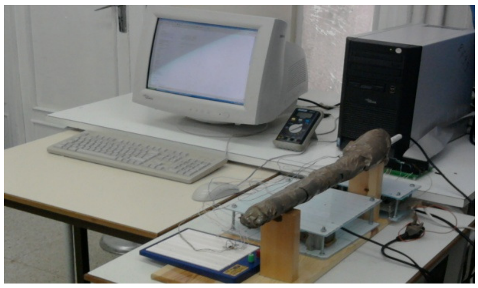

Fig.1 . The thermal system

III. PROBLEMATIC

The evolution of temperature in a metallic bar of length *L*, section *S*, conductivity λ and of a density ρ, solicited by a heat flow *J(x,t)* on the face *x*=0 is expressed by the equation of the heat Eq. 1 and generality of the Fick law Eq. 2.

$$\frac{\partial T(x,t)}{\partial t} = D\frac{\partial^2 T(x,t)}{\partial x^2} \quad (1)$$

$$J(x,t) = -\lambda S\frac{\partial T(x,t)}{\partial x} \quad (2)$$

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D is the coefficient of thermal diffusivity defined by Eq. 3:

$$D = \frac{\lambda}{\rho c} \tag{3}$$

where: c is the specific heat.

By supposing that initial condition are equal to zero and by applying the Laplace transform in the time domain to Eq.1, we can obtain the following relation.

$$pT(x,p) = D\frac{\partial^2 T(x,p)}{\partial x^2} \tag{4}$$

The general solution of Eq.4 has the following form:

$$T(x,p) = C_1(p)e^{-x\sqrt{\frac{p}{D}}} + C_2(p)e^{x\sqrt{\frac{p}{D}}} \tag{5}$$

The two terms $C_1(p)$ and $C_2(p)$ can be determined from the limit conditions (*J (L, p)* et *J (0, p))*.

Respecting the limit conditions, we can get the expression of the section impedance viewed from the input face (x=0):

$$Z(p) = \frac{T(0,p)}{J0(p)} = R_{th}\sqrt{\frac{1}{\tau p}}\frac{\left(1+e^{-2\sqrt{\tau p}}\right)}{\left(1-e^{-2\sqrt{\tau p}}\right)} = \frac{R_{th}}{\sqrt{\tau p}}\coth\sqrt{\tau p} \tag{6}$$

where:

$$R_{th} = \frac{L}{\lambda S} \qquad \tau = \frac{L^2}{D}$$

Case of important limits in practice:

- Short section $L \to 0$

If we perform a limited development to Eq.6, we can redefine the impedance Eq. 7.

$$Z(p) = \frac{1}{C_{th}p} \tag{7}$$

With C_{th} the total thermal capacity of the section defined by: $C_{th} = S\rho cL$

This limited development is the same for a given L at low frequency which means that if the system is excited with a sinusoidal signal of a low frequency, the bar with a finite length, behaves as a short length bar.

- Long section $L \to \infty$

In this case, Eq. 6 can be rewritten as follows:

$$Z(p) = \frac{r}{\sqrt{rCp}} \tag{8}$$

With r the thermal resistance with a unit of length and C the thermal capacity with a unit of length:

$$r = \frac{1}{\lambda S} \qquad C = S\rho c \tag{9}$$

The same limited development is used for a given L at high frequency which means that if the system is excited with a sinusoidal signal of high frequency, the bar of a finite length behaves same as an infinite length bar.

For the high frequencies the bar behaves like a non-integer order system equal to 0,5. The bode plot (Fig. 2) of the transfer function Eq. 6 clearly shows this frequency behavior. The corresponding parameters to aluminum are: ρ=2702 kg/m^3; c=903 J/kg.K; λ=237 w/mk

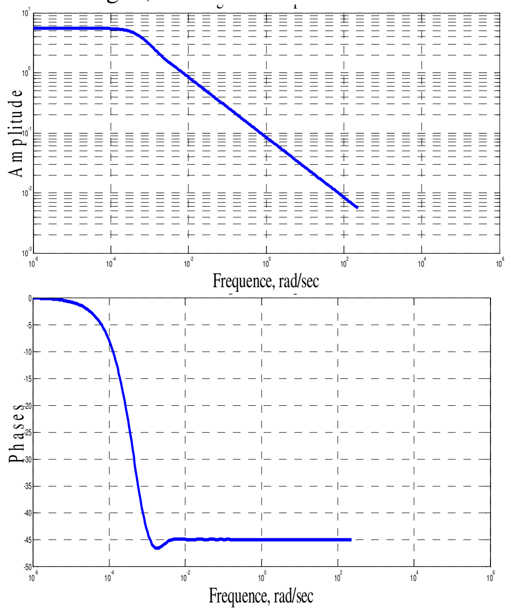

Fig. 2. Bode plot of the theory modeling of the heat transfer

All in all, heat transfer is governed by a diffusion phenomenon, which can be modeled using a non-integer operator (n=0.5). The aim of following part is to find a model able to represent diffusion behavior.

IV. MODELING OF THE DIFFUSIVE INTERFACE

A. Approach based on temporel response

To model the thermal test bench, we introduced a gain G_1 that varied from 0.14 to 0.18 (determined experimentally) according to the applied input. Using Eq. 8, the step response of the thermal test bench can be written as follows:

$$Y(p) = U(p).Z(p).G_1 = \frac{1}{p}.\frac{r}{\sqrt{rCp}}.G_1 = \frac{G_1.r}{\sqrt{rC}}\frac{1}{p^{3/2}} \tag{10}$$

The inverse Lapalace transform of $1/p^{3/2}$, can be determined based on the formula of complex integration by Eq.11:

$$TL^{-1}\left[\frac{1}{p^{3/2}}\right] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{1}{p^{3/2}}e^{pt}dp \tag{11}$$

In order to calculate this integral, we have to use the residue theorem, keeping the same closed contour of integration.

Suppose that: $F(p) = \frac{1}{p^{3/2}}e^{pt}$ (12)

The F(p) function has branch point at p=0 because of $p^{3/2}$.

The contour of integration is determined by Fig.3.

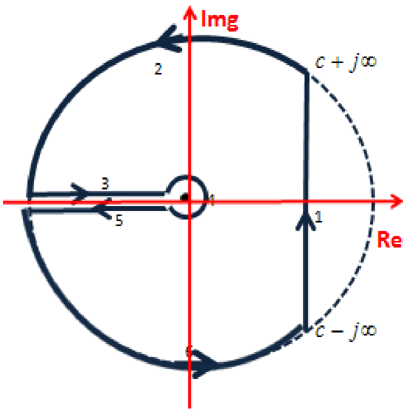

Fig.3 Contour of integration

According to the residue theorem, we can write:

$$I = \oint F(p)dp = 2\pi J \sum \text{ (Residue inside the contour)} \quad (13)$$

Finally, we get:

$$TL^{-1}\left[\frac{1}{p^{3/2}}\right] = 2\sqrt{\frac{t}{\pi}} \quad (14)$$

This leads us to write:

$$y(t) = G_1.\frac{2r}{\sqrt{rC}}\sqrt{\frac{t}{\pi}} \quad (15)$$

The bloc diagram is given by Fig. 4 , y_0 presents the ambient temperature while no input is applied to the system. The gain G_2 helps to find the temperature value, it is equal to 20 (determined experimentally).

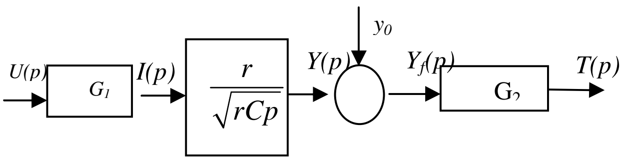

Fig.4. Functional diagram of the proposed model

The transfer functions describing the model are given with Eq.16:

$$\begin{cases} Y_f(p) = U(p)\cdot G_1.\dfrac{r}{\sqrt{rCp}} + y_0 \\ T(p) = Y_f(p).G_2 \end{cases} \quad (16)$$

The time response describing the model is given by the application of the Laplace transform to the Eq.16:

$$\begin{cases} y_f(t) = G_1\cdot\dfrac{2r}{\sqrt{rC}}\cdot\sqrt{\dfrac{t}{\pi}} + y_0 \\ T(t) = G_2.y_f(t) \end{cases} \quad (17)$$

B. Approach based on fractional order integrator

According to Trigeassou approach, the synthesis of an integrator of order n=0.5 is based on the association of an integer integrator 1/p with a fractional phase-advance filter acting in the frequency band [ω_b, ω_h]. The transfer function is given by the Eq.18.

$$I_n(p) = \frac{G_n}{p}\left(\frac{1+\dfrac{p}{\omega_b}}{1+\dfrac{p}{\omega_h}}\right)^n \equiv \frac{G_n}{p}\prod_{i=1}^{N}\frac{1+\dfrac{p}{\omega_i'}}{1+\dfrac{p}{\omega_i}} \quad (18)$$

This filter is approximated to a proposed filter by par A.Oustaloup [9], filter with advance and delay phase:

$$A_v(j\omega) = \prod_{i=1}^{N}\frac{1+j\dfrac{\omega}{\omega_i'}}{1+j\dfrac{\omega}{\omega_i}} \quad (19)$$

The non-integer integrator of order 0.5 can be synthesized from five elementary cells; the non-integer behavior shows up in the frequency band 10^{-4} and 10^{-1}. The table below shows the different values of a_i and τ_i used for the simulation.

TABLE I. Values of ai and τi

i	1	2	3	4	5
a_i	0.26	1.1	0.26	0.26	0.26
τ_i	10	100	100	1000	10000

The non-integer order integrator of 0.5 bounded in frequency is given by the equation below:

$$I_{0.5}(p) = \frac{G_{0.5}}{p}\left(\frac{1+\dfrac{p}{\omega_b}}{1+\dfrac{p}{\omega_h}}\right)^{0.5} \equiv \frac{G_{0.5}}{p}\prod_{i=1}^{5}\frac{1+\dfrac{p}{\omega_i'}}{1+\dfrac{p}{\omega_i}} \quad (20)$$

The Bode plot of the integrator is represented by Fig. 5.

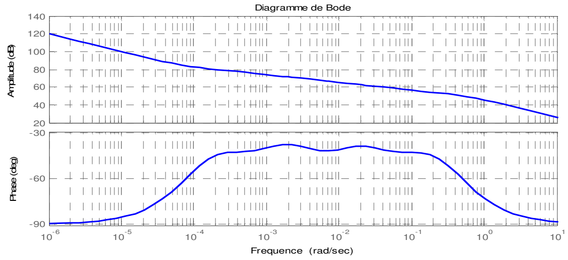

Fig. 5 Bode plot of $I_{0.5}$

The bloc diagram of the model is given by Fig. 6. The coefficient a is determined from experimental essays, it varies from 3.57 to 3.61.

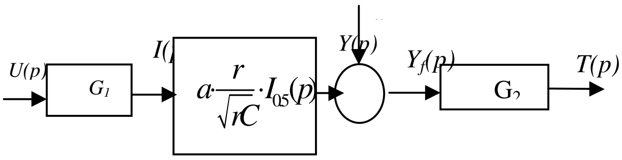

Fig. 6 Functional diagram of the poposed model

The transfer function describing the model is given by Eq. 21:

$$T(p)=\left(\left(G_1\cdot a\cdot\frac{r}{\sqrt{rC}}I_{05}(p)+y_0\right).G_2\right)U(p) \tag{21}$$

V. P EXPERIMENTAL RESULTS

We consider the system of the heat transfer with the limit conditions. Proposed models to get an approximate interface of diffusion are the first diffusive model described by Eq.16, and the second one given by Eq.21.

We apply a constant input voltage u for an hour and a half and we note the value of the temperature at 3 different positions (x_1,x_2 and x_3), with the sampling period Te= 10s. The Fig.7 shows the real step response and the theoretical one given by the proposed model based on fractional integrator.

We apply in second essay two distinctive voltages (U_1, U_2 where U_2>U_1) at x=0. Fig.8 compares the real step response with the two proposed models.

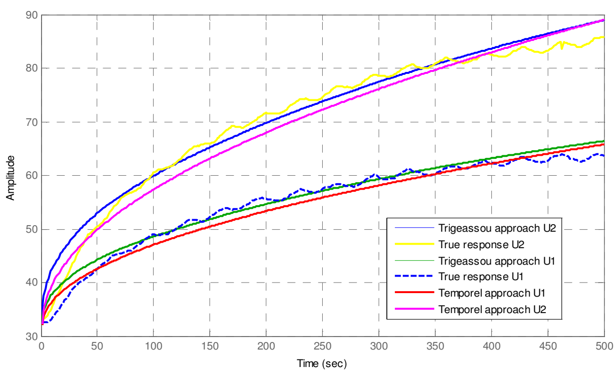

Fig.7 Response system according to two proposed model and the experimental test for an input voltage of U_1 and U_2

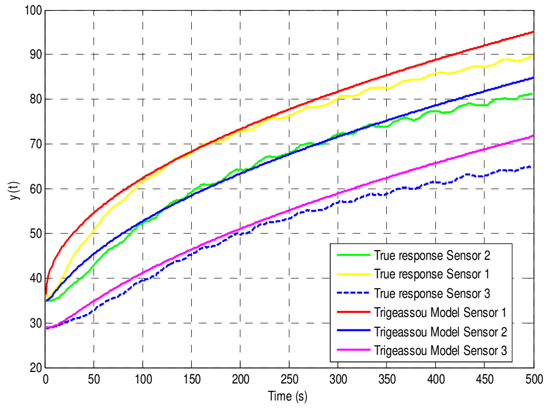

Fig.8 Response system according to Trigeassou model and the experimental test for an input voltage of U_1 and U_2 and 3 different positions

From Fig.7 and Fig.8, the temperature variation, presents waves that justifies the wave character of the heat spread proved with theory.

For both methods, the effectiveness of proposed models fitting the true response can be observed. Proposed new models of heat transfer compared to real system output give satisfactory results. It may be concluded that the two methods are very efficient in finding real behavior of heat transfer and they are also simple to implement

From Fig. 8, we can say that for the fifty first iterations, the proposed model based on temporal response is closer to the real response than the Trigeassou model. For the following iterations the Trigeassou model reproduces best the real response. Up to four hundred iterations, both models get combined.

VI. CONCLUSION

In this paper, we present two different approaches to model a thermal test bench of an aluminum heated bar at one extremity. The first model is based on temporal approach and the second one is obtained using a non-integer integrator operator where the fractional behavior acts only over a limited frequency band. Then the two different models are compared to a real diffusive system and they show effectiveness to fit the true response. Illustrative examples was presented to show the practically usefulness of the two approaches and comparative discussion was also elaborated.

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