Optimal Linear Regulator Problem with Switching Hyperplane Matrix

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Abstract— It is important to design a controller which makes the system insensitive to parameter variations and external disturbances. As it is very well known, Variable Structure Systems with Sliding Modes provides a controller which makes the system insensitive to parameter variations and external disturbances. It is also desirable that the controller should be designed in a way not to waste excessive amount of control effort. The optimal control is achieved through the minimization of a performance index in a way to regulate the states and minimizing the control effort. This problem is defined as an Optimal Linear Regulator problem. In this paper, an effort is carried out to utilize the advantages of Variable Structure System (VSS) theory' results to Continuous and Discrete-Time Optimal Control Problems. The switching hyperplane matrix parameters which provides the desired new system dynamic are used as the performance index matrix parameters. That is equivalent to minimizing the switching hyperplane where the states are forced to stay on the switching hyperplane with minimum control effort without chattering around the switching hyperplane with high frequency. The theory is extended to the Discrete and Continuous Time Stochastic Optimal Control Problems. It is shown that the application of VSS theory' results in an Optimal Control regulator problem can force the system states in a region where the system states are insensitive to plant parameter variations and external disturbances while at the same time the states are regulated in an optimal fashion with minimum control effort.

Keywords—**Variable Structure Control, Optimal Control, Continuous Time Control, Discrete Time Control, Stochastic Control.**

I. INTRODUCTION

The application of Variable Structure Systems (VSS) with Sliding mode control provides a new system dynamic being insensitive to parameter variations and external disturbances. The details of the theory is covered in many literature [1],[2],[3],[4],[13]–[17] and outlined very briefly in the following paragraphs.

Consider the state space system below;

$$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad (1)$$

$$y(t) = D\,x(t) \qquad (2)$$

Here ; x(t) is the state vector of n states, u(t) is the input vector of m inputs and y(t) is the output vector of r outputs ; A is an nxn matrix, B is an nxm matrix and D is an rxn matrix. Sliding mode control input is the input which makes the system states to move on m switching hyperplanes $s_1(t) = s_2(t) = ... = s_m(t) = 0$;

where;

$$s_j(t) = \sum_{i=1}^{n} c_{ji}\, x_i(t) = 0 \qquad (3)$$

$$j = 1, 2, ... m$$

$$i = 1, 2, ... n$$

Here ;

$$s_{jn} = 1$$

In vector notation, this is simply

$$s(t) = C\,x(t) = 0 \qquad (4)$$

Here, C is an mxn switching hyperplane matrix.

When the states move on the sliding surface simultaneously on all m switching hyperplane called as a sliding regime hyperplanes, the rate of change of the sliding surface as determined by Eq.5 should be zero in order for the states to stay on the sliding surface .

$$\frac{ds}{dt} = CA\big[x(t) + B\,u(t)\big] = 0 \qquad (5)$$

This is of course a theoretical description and the control input found by solving Eq.(5) is termed as the equivalent control the solution of which is $u_{eq}(t) = -(CB)^{-1} C\,A\,x(t)$. The equivalent control, $u_{eq}(t)$, is not the operational control applied to the plant. It is a theoretical control input which helps the designer to find the resulting sliding mode equations. If this control is substituted into Eq.(1) assuming that $(CB)^{-1}$ exists, the VSS sliding mode equations in Eq.(6) can be determined.

$$\dot{x}(t) = \big[I_n - B(CB)^{-1} C\big] A\,x(t) \qquad (6)$$

The above equations seems to be of order (nx1), however this may not be the case, since sliding surface $s(t) = C\,x(t) = 0$ is reached upon the application of the sliding mode control which in turn results m of the state variables in Eq.(3) to be defined as a linear combination of the remaining (n-m) state variables. We see that the dynamic of the system can be defined as desired by adjusting the parameters of switching hyper-plane matrix C . In this paper, the idea of forcing the states to move on a set of switching planes is tried to be applied to an optimal control problem where the switching

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hyperplane is minimized which in turn results in regulating the states as desired by adjusting the parameters of the switching hyperplane matrix , C with minimal control effort.

We know that $u_{eq}(t)$ is not the operational control applied to the system and is only helpful in determining the sliding mode equations. The operational control input applied to the plant is the so called sliding mode control in the following form [1],[2]–[3].

$$u_j(t) = -\sum_{i=1}^{n}(\alpha_j^i |x_i(t)| + \delta_j)\text{sgn}(s_j) \quad (7)$$

where

$$u_j(t) = \begin{cases} u_j^+(x(t)) & s_j(t) \geq 0 \\ u_j^-(x(t)) & s_j(t) \leq 0 \end{cases} \quad (8)$$

The control input parameters α_j^i 's are determined by considering the inequality in Eq.(9) to be satisfied so that $s(t) = 0$ is achieved .

$$\dot{s}(t)\, s(t) \prec 0 \quad (9)$$

If the above inequality is satisfied, we see that the states will reach the switching hyperplane from any initial conditions and will try to stay , chatter around the hyperplane.As a result, we say that $s(t) = 0$ is achieved. The VSS theory has also been extended to Discrete Time Systems [5],[6],[7],[8],[12],[14],[15]–[16] which is briefly summarized in the sequel.

A single input-single output discrete time system is described by the equations given below.

$$x(k+1) = A\,x(k) + b\,u(k) + d\,f(k) \quad (10)$$

where $u(k)$ is the control input and $f(k)$ is the disturbance added into the system with

$b = (0\ 0\ 0 ....b_n)$, $d = (0\ 0\ 0 ....d_n)$, A is nxn matrix and $x(k)$ is nx1 state vector.

Assume that a control input is found such that the states stay on the switching plane so that Eq.(11) is satisfied.

$$s(k) = \sum_{i=1}^{n} c_i\, x_i(k) = 0 \quad (11)$$

where c_i : are constants for $i = 1,2,.....n-1$ and . From Eq. (11), we see that $x_n(k) = -\sum_{i=1}^{n-1} c_i\, x_i(k)$

Substituting Eq.(11) into Eq.(10), the following new reduced order system equations are obtained as below,

$$x_i(k+1) = \sum_{j=1}^{n-1}(a_{ij} - c_j a_{in})\, x_j(k) \quad (12)$$

$$i = 1,2,........n-1$$

$$j = 1,2,........n-1$$

We see that the new dynamic of the system do not include the external disturbance and can be made insensitive to parameter variations by adjusting the switching hyperplane parameters appropriately. Since the system is a discrete time system, the new system dynamic which is a reduced order system is stable if the absolute values of the eigenvalues of the new system matrix are less than one. This can be achieved by adjusting the coefficients of the switching plane.

In a plant of general type, the discrete time sliding mode equations are obtained from the so called equivalent control method. The actual control applied to the system is determined as described below.

When the states reaches the sliding surface and the condition in Eq.(11) is satisfied after a number of states, the situation has to be maintained . This can be possible through a control input which is the solution of

$s_i(k+1) - s_i(k) = 0$ after having achieved $s_i(k) = 0$ (13)

If a control is found such that,

$[s_i(k+1) - s_i(k)]s_i(k) \prec 0$ and $|s_i(k+1)| \prec |s_i(k)|$ (14)

which assures both sliding motion and convergence onto the i th hyperplane which can be decomposed into two inequalities 6,[7]–[8]

$$[s_i(k+1) - s_i(k)]sign(s_i(k)) \prec 0$$
$$[s_i(k+1) + s_i(k)]sign(s_i(k)) \leq 0 \quad (15)$$

are satisfied , then the states will hit the switching hyperplane from any initial conditions and will chatter around it. As a result we say that $s(k) = 0$ is achieved.

A control input of the following form

$$u(k) = \begin{cases} u^+(x(k)) & s_i(k) \succ 0 \\ u^-(x(k)) & s_i(k) \leq 0 \end{cases} \quad (16)$$

is evaluated by taking the above conditions in Eq.16 into consideration.

In this paper, this idea of forcing the states to move on a set of switching planes is tried to be also applied to a discrete time optimal control problem where the switching plane is minimized which in turn results in regulating the states as desired by adjusting the parameters of the switching hyperplane matrix , C .

II. DISCRETE TIME LINEAR REGULATOR PROBLEM

Consider the plant

$$x(k+1) = Ax(k) + Bu(k) \quad (17)$$

with a switching hyperplane

$$s(k) = C\,x(k) \quad (18)$$

where x(k) is the state vector of n states, u(k) is the input vector of m inputs and A is an nxn matrix, B is an nxm matrix, . $s(k)$ is (mx1) vector and C is (mxn) matrix.

Define the following performance index in a similar way as described in [9] so that the switching hyperplane is minimized in an optimal fashion.

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$$J_N = \sum_{i=1}^{N} s^T(i)s(i) + u^T(i-1)R(i-1)u(i-1)) \quad (19)$$

We begin by defining V_N to be the minimum value of the performance measure J_N as follows as described in ;

$$V_N = \min_{u(0)u(1)....u(N-1)} \sum_{i=1}^{N} \left[s^T(i)s(i) + u^T(i-1)R(i-1)u(i-1)\right] \quad (20)$$

Using the principle of optimality, we proceed by starting with the last stage of control in our problem.

$$V_1 = \min_{u(N-1)} \left(s^T(N)s(N) + u^T(N-1)R(N-1)u(N-1)\right) \quad (21)$$

where

$$s(N) = Cx(N)$$
$$x(N) = Ax(N-1) + Bu(N-1) \quad (22)$$

Upon substitution in Eq.22, we obtain

$$s(N) = C\{Ax(N-1) + Bu(N-1)\}$$

Then, V_1 becomes

$$V_1 = \min_{u(N-1)} \begin{Bmatrix} (Ax(N-1) + Bu(N-1))^T C^T C \\ (Ax(N-1) + Bu(N-1)) + u^T(N-1) \\ R(N-1)u(N-1) \end{Bmatrix}$$

If we drop the time argument for simplicity,

$$V_1 = \min_{u(N-1)} (x^T A^T C^T CAx + x^T A^T C^T CBu + u^T B^T C^T CAx + u^T B^T CC^T Bu + u^T Ru)$$

Denote $Q = C^T C$, it is easily seen that Q is automatically a positive semidefinite symmetric matrix,

$$V_1 = \min_{u(N-1)} (x^T A^T QAx + x^T A^T QBu + u^T B^T QAx + u^T (B^T QB + R)u \quad (23)$$

Since Q is symmetric,

$$(u^T B^T QAx)^T = x^T A^T Q^T Bu = x^T A^T QBu$$

Then, the third term in Eq.23 is the transpose of the second term. Since both are scalars, the two terms are equal. Therefore, we write

$$V_1 = \min_{u(N-1)} (x^T A^T QAx + 2x^T A^T QBu + u^T (B^T QB + R)u) \quad (24)$$

We obtain the minimum in Eq.24 by setting the gradient of the terms with respect to u equal to zero. Then, we have

$$2x^T A^T QB + 2u^T (B^T QB + R) = 0$$

Solving for u we see that

$$u(N-1) = -(B^T QB + R)^{-1} B^T QAx(N-1) = 0$$

As it is seen, if R is selected as a positive definite matrix, the resulting control law is physically realizable and additionally is linear and involves feedback of the current state.

We define

$$L(N-1) = -(B^T QB + R)^{-1} B^T Q$$

Then,

$$u(N-1) = L(N-1)x(N-1)$$

As the reader will readily recall, in the discrete time optimal regulator problem, the following performance measure is selected [9].

$$J_N = \sum_{i=1}^{N} x^T(i)G(i)x(i) + u^T(i-1)R(i-1)u(i-1)) \quad (25)$$

For the plant in Eq.17, if we evaluate V_1 for the above J_N, it becomes

$$V_1 = \min_{u(N-1)} (x^T A^T GAx + 2x^T A^T GBu + u^T (B^T QB + R)u)$$

This is the same as Eq.24 with the exception that G is replaced by Q which is the product of the switching hyperplane matrix by its transpose. i.e CC^T. The design approach is different in our case. Rather than selecting G being at least a positive semidefinite matrix and selecting a performance index as in Eq.25 which is to regulate the states, we attempt to minimize the switching plane resulting in a new desired system dynamic and force the system states in a region where the system is less sensitive to plant parameter variations and external disturbances. If we continue to derive the performance measure for the N-stages, we get the following equations.

$$u(k) = L(k)x(k)$$
$$L(k) = -(B^T W(k+1)B + R)B^T W(k+1)A$$
$$W(k) = A^T W(k+1)A + A^T W(k+1)BL(k) + Q(k)$$ for k=N-1, N-2, ... 0.

where

$$W(N) = Q(N)$$

$B^T W(k+1)B + R$ is required to be positive definite for all k.

III. CONTINUOUS TIME LINEAR REGULATOR PROBLEM

In this section, we consider one class of optimal control problem, the linear regulator problem, in which we shall employ the VSS theory by selecting a performance measure by which the switching plane is tried to be minimized, which in turn results in state regulation in a region where there is insensitivity to parameter variations and external disturbances. Namely, the same idea followed in the previous section is followed here and applied to a continuous time linear regulator problem. The plant to be considered is described by the continuous-time linear state equations given below ;

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (26)$$
$$y(t) = Dx(t)$$

The performance measure to be minimized is selected in a similar way as described in [9] as follows ;

$$J = \frac{1}{2} s^T(t_f)s(t_f) + \frac{1}{2}\int_0^{t_f} \left[s^T(t)s(t) + u^T(t)R(t)u(t)\right]dt \quad (27)$$

The physical interpretation of this performance measure is that the switching plane is reached and minimized without an excessive expenditure of control effort, which in turn results in

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regulating the states in a region where the system is insensitive to parameter variations and external disturbances. Besides, the minimization of the switching hyperplane is achieved without chattering. One way to solve the above minimization problem is to select the Hamiltonian equations as follows;

$$H[x(t),u(t),p(t),t]=\frac{1}{2}s^T(t)s(t)+\frac{1}{2}u^T(t)R(t)u(t)+$$
$$p^T(t)A(t)x(t)+p^T(t)B(t)x(t) \qquad (28)$$

where

$$s(t)=C^T(t)x(t)$$

Then, the optimal control which minimizes the above Hamiltonian can be found by using the parallel approach defined in [9] as follows ;

$$u(t)=-R^{-1}B^T K(t)x(t) \qquad (29)$$

$$\dot{K}(t)=-K(t)A(t)-A^T K(t)-C(t)C^T(t)+$$
$$K(t)B(t)R^{-1}(t)B^T(t)K(t) \qquad (30)$$

where $K(t)$ is the gain matrix and can be evaluated as described in [9]. This is a slight modification to the original linear regulator problem and the control input evaluated by this approach facilitates the robust properties of VSS.

IV. STOCHASTIC CONTINUOUS TIME LINEAR REGULATOR PROBLEM

In the previous cases, it is assumed that all the states are available. This assumption may not be valid in practical applications. Furthermore, the processes may be stochastic. In such cases, one method is to estimate the states and use the estimated values of the states to evaluate the switching plane value, which is in fact the estimated value of the switching plane. To apply this idea, we extend the idea to the minimization of the estimate of the switching plane as described in the following paragraphs.

We consider the following system ,

$$\dot{x}(t)=A\,x(t)+B\,u(t)+G(t)w(t) \qquad (31)$$
$$y(t)=H\,x(t)+v(t)$$

The stochastic processes { $w(t), t\ge t_0$ }and { $v(t), t\ge t_0$ } are assumed to be zero mean Gaussian white noises with

$$E\left[w(t)w^T(\tau)\right]=Q(t)\,\delta(t-\tau)$$
$$E\left[v(t)v^T(\tau)\right]=R(t)\,\delta(t-\tau)$$

and

$$E\left[w(t)v^T(\tau)\right]=0$$

for all $t,\tau\ge t_0$ where all the terms have been defined previously .

If a sliding mode control which satisfies

$$\dot{s}(t)\,s(t)<0$$

where $s(t)=C\,x(t)$ as described in Eq. 7 is applied to the above system ,the system after a certain time moves on the switching plane $s(t)=0$

The equivalent control can be found as follows,

$$u_{eq}(t)=-(CB)^{-1}C\left[A\,x(t)+C\,G\,w(t)\right]$$

Upon substitution into the original system equations, Eq.31, the new dynamic is obtained as follows,

$$\dot{x}(t)=\left[I_n-B(CB)^{-1}C\right]\left[A\,x(t)+C\,G\,w(t)\right]$$

For the total disturbance rejection, the switching plane matrix must be chosen such that

$$\left[I_n-B(CB)^{-1}C\right]CG=0$$

This gives

$$\dot{x}(t)=\left[I_n-B(CB)^{-1}C\right]Ax(t) \qquad (32)$$

Then, the desired motion is achieved by adjusting the coefficients of C.

Rather than selecting a sling mode control , the switching plane may try to be minimized , which in fact enables the regulation of s(t) and achieves s(t) = 0 . Then, the system states move on the switching plane resulting in a new and desired dynamic according to the selection of the switching hyperplane with minimum control effort. To achieve this purpose, the following performance measure to be minimized is selected as follows ;

$$J=E\left\{s^T(t_f)s(t_f)+\int_{t_0}^{t_f}\left[s^T(t)s(t)+u^T(t)\Gamma(t)u(t)\right]dt\right\} \qquad (33)$$

E {s(t)} = E {s(t/t)} : Expected value of the switching hyperplane given all the measurements up to t.

If a control input is found to minimize the expected value of the above switching plane, then the expected values of the states will be forced to move in a region where we have the robust properties of the VSS, while at the same time, the estimated states are regulated in an optimal fashion. Thus, If $s(t)=C\,x(t)$ is substituted in the above performance index for s(t), the following equivalent performance index is obtained.

$$J=E\left[x^T(t_f)C^TC\,x(t_f)\right]+E\left[\int_{t_0}^{t_f}x^T(t)C^TC\,x(t)\,dt\right]$$
$$+E\left[\int_{t_0}^{t_f}u^T(t)\Gamma(t)u(t)\,dt\right] \qquad (34)$$

If the similar steps described in [10] are followed, the optimal control for the continuous stochastic linear regulator problem can be found to be characterized by the set of following relations,

$$u(t)=L(t)\,x(t/t)$$
$$L(t)=-\Gamma^{-1}B^T(t)W(t)$$
$$\dot{W}(t)=-A^TW-WA+WB\Gamma^{-1}B^T(t)W(t)-C^T(t)C(t)$$
$$\dot{x}(t/t)=Ax(t/t)+K(t)\left[z(t)-H(t)x(t/t)\right]+Bu(t)$$

It is interesting to note that in the classical VSS the measurement of the output is not taken into consideration and it is assumed that all the system states are assumed to be available. In the approach explained above, it is clear that the output measurement is taken into consideration.

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V. STOCHASTIC DISCRETE TIME LINEAR REGULATOR PROBLEM

We can extend the same argument to the discrete time stochastic linear regulator problem by following the parallel approach studied in the previous paragraphs.

We again select the following performance index for the similar reasons discussed in the previous paragraphs.

$$J_N = E\left\{\sum_{i=1}^{N} s^T(i)\,s(i) + u^T(i-1)\,R(i-1)\,u(i-1)\right\}$$

If the similar steps described in [10]-[11] are followed for a plant in Eq.17, the optimal control for the discrete time stochastic linear regulator problem can be found to be characterized by the following set of relations:

$$u(k) = L(k)\,x(k/k)$$

$$L(k) = -\left[B^T W(k+1)B + R\right]^{-1} B^T W(k+1)A$$

$$L(k) = -[B^T W(k+1) B + R]^{-1} B^T W(k+1) A$$

$W(k) = A^T W(k+1)A + A^T W(k+1)BL(k) + Q(k)$ for k=N-1, N-2, ... 0.

where

$$W(N) = Q(N)$$

$B^T W(k+1)B + R$ is required to be positive definite for all k.

$$\hat{x}(k/k) = \hat{x}(k/k-1) + K(k)\left[z(k) - H(k)\hat{x}(k/k-1\right]$$

$$\hat{x}(k/k-1) = A\hat{x}(k-1/k-1) + Bu(k-1)$$

where K(k) is the Kalman gain matrix described in [10]-[11].

V.CONCLUSION

The most favorable aspect of variable structure controller is that the new resulting system dynamic is insensitive to plant parameter variations and external disturbances. However, the control input may have a high frequency component due to the high speed chattering which is not a desired behavior and may have an effect similar to noise. Besides , the parameters of the switching hyperplane are chosen such that the states stay on the switching hyperplane and does not consider the amount of control effort .Therefore, the resulting control may be an excessive control input that may not be available due to the constraints on the control input. In this paper, the robust properties of VSS sliding mode control is extended to the conventional state regulation problem. We see that the states of the system can be regulated in an optimal fashion since we found an optimal control input which minimizes the switching hyperplane without using a sliding mode control . Here, we also remove the disadvantages of high frequency chattering when sliding mode control input is used. This approach facilitates the design of a more robust optimal control regulator by combining and facilitating the use of VSS theory. In practical applications, the system states are random. Therefore, the estimated states can only be used in the evaluation of the switching hyperplane. Therefore, the theory is also extended to cover the stochastic cases.

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